Character theory of symmetric groups

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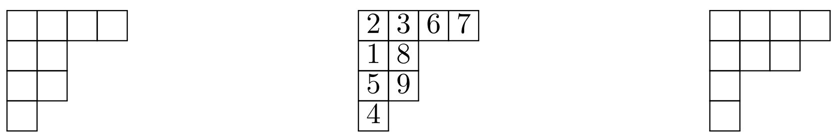
1 Ordinary characters

A *partition* of $n \in \mathbb{N}_0$ is a sequence $\lambda = (\lambda_i)_{i\in\mathbb{N}}$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \ldots$ and $|\lambda| := \sum_{i\in\mathbb{N}} \lambda_i = n$. The non-zero λ_i are called *parts* of λ, while the $\lambda_i = 0$ are usually omitted. The number of parts is called the *length* of λ. Every partition λ can be visualized with a *Young diagram* with λ_i boxes in the i-th row. By "transposing" the Young diagram (i. e. reflecting on the diagonal) we obtain the Young diagram of the *conjugate* partition $\lambda' = (\lambda'_i)$ with $\lambda'_i := |\{j : \lambda_j \geq i\}|$ for $i \in \mathbb{N}$. Obviously, $\lambda'' = \lambda$. We call λ *symmetric* if $\lambda' = \lambda$. A *Young tableau* (of λ) is a Young diagram (of λ) where every box contains exactly one of the numbers $1, \ldots, n$ and the numbers in each row are increasingly ordered.

Example 1. Let $\lambda = (4, 2, 2, 1) = (4, 2^2, 1)$ be a partition of $n = 9$. Then the Young diagram of λ, a Young tableau and the conjugate Young diagram are given by:



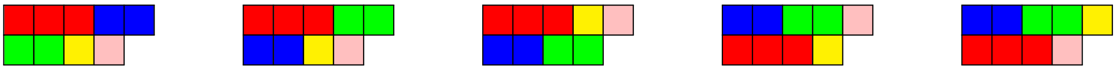
Every conjugacy class of the symmetric group S_n consists of the elements with a common cycle type. Therefore, the conjugacy classes of S_n can be identified with the partitions of n and $\mathrm{sgn}(\lambda) = (-1)^{n-l}$ makes sense for partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n. The Young tableaux of λ are in one-to-one correspondence with the (ordered) partitions $Y = (Y_1, Y_2, \ldots)$ of the set $\{1, \ldots, n\}$ such that $|Y_i| = \lambda_i$ for $i \in \mathbb{N}$. Hence, S_n acts transitively on the set of Young tableaux of λ via ${}^gY = ({}^gY_i)$ for $g \in S_n$. The stabilizer of Y is the *Young subgroup* $S_Y := \prod \mathrm{Sym}(Y_i) \leq S_n$ and the permutation character is $\psi_\lambda := (1_{S_Y})^{S_n}$. The characters ψ_λ and $\mathrm{sgn}\,\psi_{\lambda'}$ (where sgn is the *sign* character) have exactly one irreducible constituent χ_λ. Then $\chi_{\lambda'} = \mathrm{sgn}\,\chi_\lambda$ and

$$\mathrm{Irr}(S_n) = \{\chi_\lambda : \lambda \text{ partition of } n\}.$$

Example 2. We have $\psi_{(n)} = 1_{S_n} = \chi_{(n)}$ and $\chi_{(1^n)} = \chi_{(n)'} = \mathrm{sgn}$. The Young tableaux of $(n-1, 1)$ can be identified with the numbers $1, \ldots, n$. Hence, $\psi_{(n-1,1)}$ is the natural (2-transitive) permutation character of S_n and $\chi_{(n-1,1)} = \psi_{(n-1,1)} - 1_{S_n}$ for $n \geq 2$.

Let λ and μ be partitions of n. If $g \in S_n$ has type μ, then $\psi_\lambda(g)$ is the number of ways to distribute the parts of μ onto the parts of λ.

Example 3. For $\lambda = (5,4)$ and $\mu = (3,2^2,1^2)$, we obtain $\psi_\lambda(g) = 5$ as follows:



Starting with $\psi_{(n)} = \chi_{(n)} = 1_{S_n}$, one can compute $\mathrm{Irr}(S_n)$ recursively via

$$\chi_\lambda = \psi_\lambda - \sum_{\mu>\lambda} [\psi_\lambda, \chi_\mu]\chi_\mu = \psi_\lambda - 1_{S_n} - \sum_{(n)>\mu>\lambda} [\psi_\lambda, \chi_\mu]\chi_\mu$$

where $>$ denotes the lexicographical order. In fact, χ_μ can only occur in ψ_λ if $\mu \trianglerighteq \lambda$, i. e.

$$\sum_{i=1}^{s} \mu_i \ge \sum_{i=1}^{s} \lambda_i \qquad (s = 1, 2, \ldots)$$

(*dominance order*).

The *hook* $h_{ij}(\lambda) = h_{ij}$ of a box (i,j) of the Young diagram Y of a partition λ is the union of the boxes (i,j), $(i,j+1)$, $\ldots$ and the boxes $(i+1,j)$, $(i+2,j)$, $\ldots$. Then $|h_{ij}| = \lambda_i + \lambda'_j - i - j + 1$ is the *hook length* and the *hook length formula* holds

$$\chi_\lambda(1) = \frac{n!}{\prod\limits_{(i,j) \text{ box of } Y} |h_{ij}|}.$$

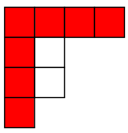
Let t_k be the number of k-cycles of some $g \in S_n$. *Frobenius' character formula* states that $\chi_\lambda(g)$ is the coefficient of $X_1^{h_{11}} X_2^{h_{21}} \ldots$ in the polynomial

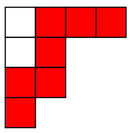
$$\prod_{i<j} (X_i - X_j) \prod_{k \ge 1} (X_1^k + X_2^k + \ldots)^{t_k}.$$

Let $l_{ij} := \lambda'_j - i$ (resp. $a_{ij} := \lambda_i - j$) be the *leg length* (resp. *arm length*). Removing h_{ij} from Y yields a Young diagram of a partition $\lambda \setminus h_{ij}$ of $n - |h_{ij}|$. Equivalently, one can remove the corresponding *rim hook*.

Example 4. A Young diagram filled with hook lengths, the hook h_{11} and its rim hook:

7	5	2	1
4	2		
3	1		
1			





Next, let $g \in S_n$ of type μ and let $h \in S_{n-\mu_k}$ be of type $(\mu_1, \ldots, \mu_{k-1}, \mu_{k+1}, \ldots)$. Let Y be the Young tableau of λ. Then the *Murnaghan-Nakayama formula* states that

$$\chi_\lambda(g) = \sum_{\substack{(i,j) \text{ box of } Y \\ |h_{ij}| = \mu_k}} (-1)^{l_{ij}} \chi_{\lambda \setminus h_{ij}}(h).$$

The special case $\mu_k = 1$ is called *branching rule*

$$(\chi_\lambda)_{S_{n-1}} = \sum_{\substack{(i,j) \text{ box of } Y \\ |h_{ij}| = 1}} \chi_{\lambda \setminus h_{ij}}.$$

2 Specht modules

Let $T_1, \ldots, T_k$ be the Young tableaux of a given partition λ of n. Note that $k = \frac{n!}{\prod \lambda_i!}$. The $\mathbb{Q}$-vector space M with basis $T_1, \ldots, T_k$ is the $\mathbb{Q}S_n$-permutation module with character ψ_λ as defined above. Let Y_i' be the set partition of $\{1, \ldots, n\}$ corresponding to the conjugate tableau T_i' of λ'. The *Specht module* S^λ associated with λ is the submodule of M generated by the elements

$$t_i := \sum_{\pi \in S_{Y_i'}} \operatorname{sgn}(\pi)^\pi T_i \qquad (i = 1, \ldots, k)$$

(it is easy to see that ${}^\pi T_i \neq {}^\sigma T_i$ for $\pi \neq \sigma$). It turns out that S^λ is simple with character χ_λ. In particular, all irreducible characters of S_n can be realized over $\mathbb{Z}$. Therefore, the Frobenius-Schur indicators are always 1. A basis of S^λ is given by those t_i such that T_i is *standard*, i. e. also the columns of T_i are increasingly ordered. Thus, the hook formula also counts the number of standard Young tableaux of λ.

3 Blocks

Let p be a prime. A *p-hook* is a hook of length p. Starting from a partition λ we can successively remove all p-hooks from the corresponding Young diagram to obtain the *p-core* which is a partition of $n - wp$ where w is the *weight* of λ (this does not depend on the way the hooks are removed). Characters $\chi_\lambda, \chi_\mu \in \operatorname{Irr}(S_n)$ lie in the same *p-block* if and only if they have the same p-core (*Nakayama's conjecture*). In this way, the p-blocks of S_n can be labeled by p-cores. The *weight* of a block B is the weight of any λ with $\chi_\lambda \in \operatorname{Irr}(B)$. Note that conjugate characters (and blocks) have conjugate cores. The principal block containing $1_{S_n} = \chi_{(n)}$ corresponds to the core (r) where $r \in \{0, \ldots, p-1\}$ such that $n \equiv r \pmod p$. The blocks of weight 0 contain only one irreducible character χ_λ where λ is a core. By the hook formula, $|S_n|_p = \chi_\lambda(1)_p$. Hence, these are the blocks of p-defect 0. Ono proved that p-defect 0 characters exist for all n and $p \geq 5$. Note that the 2-cores are the *staircase* partitions $(k, k-1, \ldots, 1)$. In particular, S_n has at most one 2-block of weight w and in that case $n - 2w = \binom{k+1}{2}$ is a triangular number.

In general, the *fusion system* of a p-block B of weight w is the fusion system of S_{pw} with respect to its Sylow p-subgroup P of order $p^{w+\lfloor w/p \rfloor + \cdots}$ (Legendre's formula). In particular, P is a defect group of B. If $w = \sum a_i p^{i-1}$ is the p-adic expansion (i. e. $0 \leq a_i \ldots < p$), then $P \cong \prod P_i^{a_i}$ where $P_i := C_p \wr \ldots \wr C_p$ (i copies).

4 Equivalences

Enguehard has shown that two p-blocks of (possibly different) symmetric groups with the same weight w are *perfectly isometric*. It was later shown that each such block B is *splendid derived equivalent* to the principal block of S_{wp}. In particular,

$$k(B) := |\operatorname{Irr}(B)| = \sum_{\substack{(w_1, \ldots, w_p) \in \mathbb{N}_0^p \\ \sum w_i = w}} \pi(w_1) \ldots \pi(w_p)$$

where $\pi(m)$ is the number of partitions of $m \in \mathbb{N}_0$. Obviously, P is abelian if and only if $w < p$ and in this case *Broué's conjecture* holds.

The *(p)-abacus* $A_\lambda \subseteq \{0,\ldots,p-1\}\times\mathbb{N}_0$ of a partition λ is defined by $(r,s)\in A_\lambda \Leftrightarrow \exists i : r+sp = h_{i1}$. The elements of A_λ can be visualized as *beads* on a matrix with infinitely many columns. The rows of this matrix are called *runners*. Removing a box from the Young diagram of λ is the same as moving a bead of A_λ up to the previous runner (modulo p). Removing a p-hook slides a bead to the left by one (in particular this spot must be vacant beforehand). Hence, the abacus of a core has no "holes" and its first runner is empty.

Let B be a block of weight w with core μ. Let a_i be the number of beads on runner i of A_μ. Suppose that $a_{i+1}-a_i \geq w$ for some $i\in\{0,\ldots,p-2\}$. Then, interchanging runner i and $i+1$ yields a core of a block $\hat{B}$ of $S_{n-a_{i+1}+a_i}$ which is Morita equivalent to B (*Scopes' reduction*). Thus, in order to determine the Morita equivalence class of B we may assume that $a_{i+1}-a_i < w$ for $i=0,\ldots,p-2$. Since $a_0=0$, it follows that $a_i \leq i(w-1)$ for all i. The number of blocks with these restrictions is $\frac{1}{p}\binom{wp}{p-1}$. If $\mu\neq\mu'$, then B is also Morita equivalent to the block B' of S_n with core μ' (note that $\mathrm{Irr}(B')=\operatorname{sgn}\mathrm{Irr}(B)$). Therefore the number of Morita equivalence classes of p-blocks of symmetric groups of weight w is at most

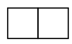
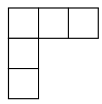
$$\frac{1}{2p}\binom{wp}{p-1}+\frac{1}{2}\binom{\lfloor wp/2\rfloor}{\lfloor p/2\rfloor}.$$

If $a_i = i(w-1)$ for $i=0,\ldots,p-1$, then B is called *RoCK block* and

$$n=\frac{p}{24}\Big((w-1)^2p(p^2-1)+2(w-1)p^2+22w+2\Big).$$

In the case $w<p$ the RoCK is Morita equivalent to its Brauer correspondent in $\mathrm{N}_{S_n}(D)$ where D is a defect group of B. Moreover, B is Morita equivalent to the principal block of $S_p \wr S_w$.

Example 5. The Morita equivalence classes of 3-blocks of S_n of weight (defect) 2 are represented by the principal blocks of S_6, S_7 and a non-principal block of S_{11}. The cores and abaci are given as follows:

empty abacus/core $\quad \begin{array}{c|c} 0 & \cdot \\ 1 & \bullet \\ 2 & \cdot \end{array}$  $\quad \begin{array}{c|c} 0 & \cdot \\ 1 & \cdot \\ 2 & \bullet \end{array}$  $\quad \begin{array}{c|cc} 0 & \cdot & \cdot \\ 1 & \bullet & \cdot \\ 2 & \bullet & \bullet \end{array}$ 

5 Decomposition numbers

In general, the number of irreducible Brauer characters of a finite group equals the number of conjugacy classes of p-regular elements. For S_n this is the number of partitions with no non-zero part divisible by p. A partition is called *p-regular* if it has no p parts of the same non-zero length (for $p=2$ this means that all parts are distinct). By *Glaisher's Theorem*, also the number of these partitions is the number of irreducible Brauer characters (for $p=2$ this is *Euler's Theorem*: the number of partitions with distinct parts is the number of partitions with odd parts). Starting from an arbitrary partition λ we construct a p-regular partition λ^0 by successively removing p-hooks with arm length 0. For a p-block B with weight w and core μ the number of irreducible Brauer characters in B equals the number of p-regular partitions with core μ. We write $\mathrm{IBr}(B):=\{\varphi_\lambda : \chi_\lambda\in\mathrm{Irr}(B),\ \lambda^0=\lambda\}$ and

$$l(B):=|\mathrm{IBr}(B)|=\sum_{\substack{(w_1,\ldots,w_{p-1})\in\mathbb{N}_0^{p-1}\\ \sum w_i=w}}\pi(w_1)\ldots\pi(w_{p-1}).$$

Unlike in the ordinary case there is no formula for the degrees of Brauer characters. In fact, for $p=2$ and $n\geq 20$ (say) these degrees are unknown. We denote the decomposition numbers of B by $d_{\lambda\tau}:=d_{\chi_\lambda\varphi_\tau}$.

If the irreducible characters of B are ordered in such a way that the p-regular partitions in decreasing lexicographical order come first, then the decomposition matrix $(d_{\lambda\tau})$ has unitriangular shape.

For partitions λ and μ of n let

$$t_{\lambda\mu} := -\sum_{\lambda\setminus h_{ij}(\lambda)=\mu\setminus h_{kl}(\mu)} (-1)^{l_{ij}(\lambda)+l_{kl}(\mu)}\nu_p(|h_{ij}(\lambda)|)$$

where ν_p is the p-adic valuation. Then the *Jantzen-Schaper formula* states that

$$d_{\lambda\tau} \le \sum_{\mu>\lambda} t_{\lambda\mu}d_{\mu\tau}$$

for $\lambda \ne \tau$. Moreover, $d_{\lambda\tau} = 0$ if and only if the right hand side is 0. For blocks of weight at most 3 it is known that $d_{\lambda\tau} \le 1$ and therefore $(d_{\lambda\tau})$ can be computed recursively. More explicit results for $w = 2$ and $w = 3$ were given by Richards and Fayers respectively.

6 Cartan invariants

We have seen above that $k(B)$ and $l(B)$ only depend on the weight w of a block B of S_n. We therefore write $k(w) := k(B)$ and $l(w) := l(B)$. The elementary divisors of the Cartan matrix $C(B)$ of B will also depend solely on w (but $C(B)$ itself depends on more than that). We make use of the generating function $P(x) := \sum_{k\ge 0}\pi(k)x^k$. A formula of Euler states that

$$P(x) = \prod_{k=1}^{\infty}\frac{1}{1-x^k}.$$

Moreover, if $\pi_0(n)$ is the number of p-regular partitions of n, then

$$\sum_{n\ge 0}\pi_0(n)x^n = P(x)P(x^p)^{-1}.$$

The results above can be rephrased as

$$\sum_{w\ge 0} k(w)x^w = P(x)^p, \tag{6.1}$$

$$\sum_{w\ge 0} l(w)x^w = P(x)^{p-1}. \tag{6.2}$$

Let $m(w)$ be the multiplicity of 1 as an elementary divisor of $C(B)$. Then

$$\sum_{w\ge 0} m(w)x^w = P(x)^{p-2}P(x^p).$$

In particular, $m(w) > 0$ if $p > 2$ and

$$m(w) = \begin{cases}\pi(w/2) & \text{if } w \equiv 0 \pmod 2\\ 0 & \text{otherwise}\end{cases}$$

if $p = 2$. For a partition $\lambda = (\lambda_1, \ldots)$ let

$$e(\lambda) = \sum_{k\ge 1}\frac{p^{\nu_p(\lambda_k)+1}-1}{p-1}.$$

Let $\pi_0^e(n)$ be the number of p-regular partitions λ of n such that $e(\lambda) = e$. A theorem of Olsson says that the multiplicity of p^e as an elementary divisor of $C(B)$ is

$$\sum_{s=0}^{w} m(w-s)\pi_0^e(s).$$

It is also possible to express the multiplicities of lower defect groups of B.

Example 6. The principal 2-block B of S_{10} has weight $w = 5$. We only need the 2-regular partitions of 1, 3, 5:

λ	(1)	$(2,1)$	(3)	$(3,2)$	$(4,1)$	(5)
$e(\lambda)$	1	4	1	4	8	1

Hence, 2^e can only occur as elementary divisor if $e \in \{1, 4, 8\}$. The multiplicity of $2^8 = |D|$ is always 1. The multiplicities of 2 and 16 are

$$m(4)\pi_0^1(1) + m(2)\pi_0^1(3) + m(0)\pi_0^1(5) = 2 + 1 + 1 = 4,$$
$$\pi_0^4(3) + \pi_0^4(5) = 1 + 1 = 2$$

respectively. In general, the multiplicity of 2 is $\pi(0) + \ldots + \pi(k)$ if $w = 2k + 1$ and 0 otherwise.

7 Heights

Let $n = \sum a_i p^i$ is the p-adic expansion where p is a prime. For any expansion $n = \sum b_i p^i$ with $b_0, b_1, \ldots \geq 0$ let

$$\delta(b_0, b_1, \ldots) := \frac{\sum_i b_i - a_i}{p-1} \geq 0.$$

Let $E_d(n)$ be the set of those sequences $(b_0, \ldots)$ such that $\delta(b_0, \ldots) = d$.

Next let $c(n)$ be the number of p-core partitions of n (= number of blocks of defect 0 of S_n). Set $C(x) := \sum_{n \geq 0} c(n)x^n$. Generalizing (6.1) we define

$$P(x)^s = \sum_{t=0}^{\infty} k(s,t)x^t,$$
$$C(x)^s = \sum_{t=0}^{\infty} c(s,t)x^t.$$

Note that if $t < p$, then $c(t) = \pi(t)$ and $c(s,t) = k(s,t)$ for all $s \geq 0$. Let $m_d(n)$ be the number of $\chi \in \mathrm{Irr}(S_n)$ such that $\chi(1)_p = p^d$. Olsson has shown that

$$m_d(n) = \sum_{(b_0,\ldots) \in E_d(n)} c(1, b_0)c(p, b_1)c(p^2, b_2) \ldots.$$

For $d = 0$ we have $E_0(n) = \{(a_0, \ldots)\}$ and this yields *MacDonald's Theorem*

$$m_0(n) = k(1, a_0)k(p, a_1) \ldots.$$

If additionally $p = 2$, then $a_i \leq 1$ and $m_0(n) = 2^{a_0 + \cdots}$. In particular, if $n = 2^k$, then $m_0(n) = n$ and the corresponding characters $\chi_\lambda \in \mathrm{Irr}(S_n)$ (of odd degree) are labeled by the *hook partitions* $\lambda = (s, 1^{n-s})$ for $s = 1, \ldots, n$.

Now let B be a p-block of S_n with weight w and defect d. The *height* $h(\chi) \ge 0$ of $\chi \in \operatorname{Irr}(B)$ is defined by $\chi(1)_p p^{d-h(\chi)} = |S_n|_p$. Let $k_h(w)$ be the number of $\chi \in \operatorname{Irr}(B)$ of height h (depends only on w). Then

$$k_h(w) = \sum_{(b_0,\ldots)\in E_h(w)} c(p,b_0)c(p^2,b_1)\ldots.$$

Since for $n = pw$ there is only one block of maximal defect in S_n, we recover $k_0(w) = m_0(pw)$. The maximal possible height of some $\chi \in \operatorname{Irr}(B)$ is

$$h = \frac{w - \sum a_i}{p-1}$$

where $w = \sum a_i p^i$ is the p-adic expansion. Then $k_h(w) = c(p,w)$ since $E_h(w) = \{(w,0,\ldots)\}$. For $p = 2$ it can happen that $k_h(w) = 0$ (e. g. $k_3(5) = c(2,5) = 0$). In general, Olsson's Conjecture $k_0(w) \le |D : D'|$ holds where D is a defect group of B.

Example 7. For $p = 2$ and $n = 7 = 1 + 2 + 4$ we have $(a_0, a_1, a_2) = (1, 1, 1)$ and

$$E_1(7) = \{(3,0,1),(1,3)\}, \qquad E_2(7) = \{(3,2)\}, \qquad E_3(7) = \{(5,1)\}, \qquad E_4(7) = \{(7)\}.$$

Moreover,

$$C(x) = 1 + x + x^3 + x^6 + x^{10} + \ldots, \qquad C(x)^2 = 1 + 2x + x^2 + 2x^3 + \ldots, \qquad C(x)^4 = 1 + 4x + \ldots$$

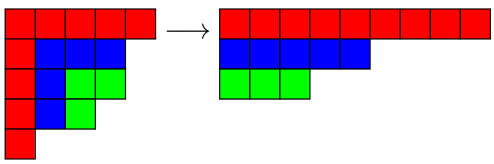
Consequently,

$$\begin{aligned}
m_0(7) &= 2^{1+1+1} = 8,\\
m_1(7) &= c(1,3)c(4,1) + c(1,1)c(2,3) = 4 + 2 = 6,\\
m_2(7) &= c(1,3)c(2,2) = 1,\\
m_3(7) &= c(1,5)c(2,1) = 0,\\
m_4(7) &= c(1,7) = 0
\end{aligned}$$

8 Alternating groups

A conjugacy class C of A_n lies in a conjugacy class of S_n and therefore belongs to a partition λ of n. More precisely, C is not a conjugacy class of S_n if and only if λ has distinct odd parts. In this case $C \,\dot\cup\, C^{(12)}$ is a conjugacy class of S_n. By *Sylvester's Theorem* there is a bijection Γ between the symmetric partitions and the partitions with distinct odd parts:

$$(\lambda_1, \lambda_2, \ldots) \xrightarrow{\Gamma} (2\lambda_1 - 1, 2\lambda_2 - 3, \ldots)$$



If $\lambda \ne \lambda'$, then $(\chi_\lambda)_{A_n} \in \operatorname{Irr}(A_n)$. Now suppose that $\lambda = \lambda'$ and $\mu := \Gamma(\lambda)$. Then by Clifford theory, $(\chi_\lambda)_{A_n} = \xi_\lambda + \xi_\lambda^{(12)}$ for some $\xi_\lambda \in \operatorname{Irr}(A_n)$ with $\xi_\lambda^{S_n} = \chi_\lambda$. We fix $g \in A_n$ of type μ. Then for $h \in A_n$

and $\epsilon := (-1)^{\frac{n-l(\mu)}{2}}$ we have

$$\xi_\lambda(h) = \begin{cases} \frac{1}{2}\chi_\lambda(h) & \text{if } h \text{ is not of type } \mu, \\ \frac{1}{2}\big(\epsilon + \sqrt{\epsilon\mu_1 \ldots \mu_{l(\mu)}}\big) & \text{if } h \text{ is conjugate to } g \text{ in } A_n, \\ \frac{1}{2}\big(\epsilon - \sqrt{\epsilon\mu_1 \ldots \mu_{l(\mu)}}\big) & \text{if } h \text{ is conjugate to } g^{(12)} \text{ in } A_n. \end{cases}$$

This allows to compute the character table of A_n from $\mathrm{Irr}(S_n)$.

Similarly, if B is a p-block of S_n with core $\mu \neq \mu'$, then B is isomorphic to a block B' of A_n via restriction. In this case, $p > 2$ and B and B' have the same fusion system. Now suppose that B has core $\mu = \mu'$, weight w and defect group D. Then B covers a block B' of A_n with defect group $D \cap A_n$ and fusion system A_{wp}. Let C' be p-block of A_m obtain in the same way by some core $\nu = \nu'$ such that μ and ν have the same weight. Brunat–Gramain have shown that B' and C' are perfectly isometric. If $p = 2$, every core has the form $\mu = (a, a-1, \ldots, 1) = \mu'$. If in addition w is odd, then every $\chi \in \mathrm{Irr}(B)$ restricts to $\mathrm{Irr}(B')$. Hence, in this case, $k(B) = 2k(B')$ and the decomposition matrix of B consists of two copies of the decomposition matrix of B'. Marcus has proved that every p-block of A_n with abelian defect group is splendid derived equivalent to its Brauer correspondent (Broué's conjecture). For an odd prime p let $p^* = (-1)^{\frac{p-1}{2}}p$. Robinson and Thompson have shown that if $n \geq 25$, then

$$\mathbb{Q}(A_n) = \mathbb{Q}(\sqrt{p^*} : 3 \leq p \leq n \text{ prime }, p \neq n-2).$$

9 Wreath products

Generalizing the abacus we call any strictly decreasing sequence $a = (a_i) \in \mathbb{N}_0^l$ a *β-set* of *length* $l(a) = l$. We often identify β-sets with finite subsets of $\mathbb{N}_0$. For $s \in \mathbb{N}_0$ also

$$a^{+s} := (a_1 + s, \ldots, a_l + s, s-1, s-2, \ldots, 0)$$

is a β-set (of length $l+s$). Any β-set a determines a partition $\lambda := P(a) := (a_1-(l-1), a_2-(l-2), \ldots, a_l)$ (note that a is the set of first column hook lengths of λ). Since $P(a) = P(a^{+s})$, we may assume that $l(a) \equiv 0 \pmod p$ in the following. We define $a_i^{(p)} := \{b \in \mathbb{N}_0 : bp + i \in a\}$ for $i = 0, \ldots, p-1$ (that is, we look at each runner of the abacus individually). Then the sequence of partitions $\lambda^{(p)} := (P(a_0^{(p)}), \ldots, P(a_{p-1}^{(p)}))$ is called the *p-quotient* of λ. The number $\sum |P(a_i^{(p)})|$ equals the weight of λ. Conversely, λ is uniquely determined by its p-core and p-quotient. If μ is the p-core of λ, the *p-sign* of λ is defined by $\delta_p(\lambda) = (-1)^{\sum l_i}$ where the l_i are the leg lengths of the p-hooks removed from λ to obtain μ.

Example 8. For $\lambda = (5, 4, 1^2)$ and $p = 2$ we obtain

$$a = (8, 6, 2, 1), \qquad (a_i^{(p)}) = (\{4, 3, 1\}, \{0\}), \qquad \lambda^{(p)} = ((2, 2, 1), ()).$$

Hence, λ has weight 5 and the p-core is (1).

Let B be a p-block of S_n with weight w. Let $\mathrm{Irr}(C_p) = \{\varphi_1, \ldots, \varphi_p\}$ and let $\tau = (\tau_1, \ldots, \tau_p)$ a tuple of partitions such that $\sum |\tau_i| = w$. The linear characters $\varphi^{\otimes|\tau_i|} := \varphi_i \otimes \ldots \otimes \varphi_i \in \mathrm{Irr}(C_p^{|\tau_i|})$ extend to $C_p \wr S_{|\tau_i|}$ and we can define $\varphi_{\tau_i} := \varphi^{\otimes|\tau_i|}\chi_{\tau_i} \in \mathrm{Irr}(C_p \wr S_{|\tau_i|})$ where $\chi_{\tau_i} \in \mathrm{Irr}(S_{|\tau_i|})$. Finally let $\varphi_\tau := \big(\bigotimes_{i=1}^p \varphi_{\tau_i}\big)^{C_p \wr S_w} \in \mathrm{Irr}(C_p \wr S_w)$. Then $\mathrm{Irr}(B) \to \mathrm{Irr}(C_p \wr S_w)$, $\chi_\lambda \mapsto \varphi_{\lambda^{(p)}}$ is a height preserving bijection.

Now we label the conjugacy classes of $C_p \wr S_w$ where we consider C_p as $\mathbb{Z}/p\mathbb{Z}$. For $(x_1 \ldots x_w, \sigma) \in C_p \wr S_w$ we define a tuple of partitions $\tau = (\tau_0, \ldots, \tau_{p-1})$ as follows: For every cycle $(a_1, \ldots, a_s)$ in σ let $s \in \tau_{x_{a_1}+\ldots+x_{a_s}}$. Then $\sum |\tau_i| = w$. Let $g_1, \ldots, g_l \in C_p \wr S_w$ be representatives for the classes of $C_p \wr S_w$ corresponding to the partition tuples τ with $\tau_0 = ()$ (note that these elements are non-trivial). Osima has shown that there exists $S \in \mathrm{GL}(l(B), \mathbb{C})$ such that

$$(d_{\chi_\lambda, i}) = (\delta_p(\lambda)\varphi_{\lambda^{(p)}}(g_i))S$$

where $(d_{\chi_\lambda,i})_{\lambda,i}$ is the decomposition matrix of B. It follows that the so-called *contributions* of B can be computed inside the smaller group $C_p \wr S_w$. More precisely,

$$[\chi_\lambda, \chi_\mu]^0 = \frac{1}{n!}\sum_{g \in S_n^0} \chi_\lambda(g)\chi_\mu(g^{-1}) = \delta_p(\lambda)\delta_p(\mu)\sum_{i=1}^{l} \frac{1}{|\mathrm{C}_{C_p \wr S_w}(g_i)|}\varphi_{\lambda^{(p)}}(g_i)\varphi_{\mu^{(p)}}(g_i^{-1})$$

for every $\chi_\lambda, \chi_\mu \in \mathrm{Irr}(B)$.

10 Double covers and spin blocks

The Schur multiplier $M(S_n) := H^2(S_n, \mathbb{C}^\times)$ is trivial for $n \le 3$ and of order 2 for $n \ge 4$. For $4 \le n \ne 6$ there are two non-isomorphic double covers:

$$\widehat{S}_n := \langle x_1, \ldots, x_{n-1}, z \mid z^2 = 1, x_i^2 = (x_i x_{i+1})^3 = [x_i, x_j] = z \text{ for } i < j-1\rangle,$$
$$\widetilde{S}_n := \langle x_1, \ldots, x_{n-1}, z \mid z^2 = 1, x_i^2 = (x_i x_{i+1})^3 = 1, [x_i, x_j] = z \text{ for } i < j-1\rangle$$

with $\mathrm{Z}(\widehat{S}_n) = \mathrm{Z}(\widetilde{S}_n) = \langle z\rangle$ (some authors swap $\widehat{S}_n$ and $\widetilde{S}_n$). The outer automorphism of S_6 induces an isomorphism $\widehat{S}_6 \cong \widetilde{S}_6$. In the following we assume $n \ge 4$. For $g \in \widehat{S}_n$ let $\overline{g} := g\langle z\rangle \in S_n$. The cycle type μ of $\overline{g}$ is called the *type* of g. One can show that g and gz are not conjugate in $\widehat{S}_n$ if and only if μ has only odd parts or (μ has distinct parts and $\mathrm{sgn}(\mu) = -1$). In this case one can choose a canonical representative g for a given μ in terms of the presentation of $\widehat{S}_n$ above. We omit the details.

We regard $\mathrm{Irr}(S_n)$ as a subset of $\mathrm{Irr}(\widehat{S}_n)$ by inflation. The characters in $\mathrm{Irr}(\widehat{S}_n) \setminus \mathrm{Irr}(S_n)$ are called *spin characters* (these are the faithful characters of $\widehat{S}_n$). They correspond to the projective characters of S_n. The partitions of n with pairwise distinct parts are called *bar partitions* (this is the same as 2-regular). For each bar partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ of n we can choose a spin character $\hat{\chi}_\lambda$ such that $\hat{\chi}_\lambda \ne \hat{\chi}_\mu$ for $\lambda \ne \mu$. Moreover, $\hat{\chi}_\lambda = \mathrm{sgn}\,\hat{\chi}_\lambda$ if and only if $\mathrm{sgn}(\lambda) = 1$ (i.e. $n \equiv l \pmod 2$ where l is the length of λ). The characters $\hat{\chi}_\lambda$ and $\mathrm{sgn}\,\hat{\chi}_\lambda$ (if $\mathrm{sgn}(\lambda) = -1$) constitute all spin characters. For the canonical representative $g \in \widehat{S}_n$ of type μ we have

$$\hat{\chi}_\lambda(g) = -\hat{\chi}_\lambda(gz) = \begin{cases} \sqrt{-1}^{\frac{n-l+1}{2}}\sqrt{\lambda_1 \ldots \lambda_l/2} & \text{if } \tau = \lambda \text{ and } \mathrm{sgn}(\lambda) = -1, \\ 0 & \text{if } \tau \ne \lambda \text{ has an even part} \end{cases}$$

(note that $\lambda_1 \ldots \lambda_l/2$ is an integer for $\mathrm{sgn}(\lambda) = -1$). Now assume that $\mu \ne \lambda$ has only odd parts. Let r be a fixed part of μ and let $h \in \widehat{S}_{n-r}$ such that $h\,\mathrm{Z}(\widehat{S}_{n-r}) \in S_{n-r}$ has cycle type $\tau \setminus r$. We may remove r also from λ in one of the following ways:

- If some $\lambda_i > r$ and $\lambda_i - r$ is not in λ, we obtain $\lambda \setminus_i r$ by replacing λ_i with $\lambda_i - r$. Moreover, define $\epsilon_i := (-1)^{j-i}2^{\frac{\mathrm{sgn}(\lambda)+1}{2}}$ where $\lambda_j > \lambda_i - r > \lambda_{j+1}$.
- If some $\lambda_i = r$, we obtain $\lambda \setminus_i r$ by removing λ_i from λ. Define $\epsilon_i := (-1)^{l-i}$.

- If $r = \lambda_i + \lambda_j$ for some $i < j$, we obtain $\lambda \backslash_i r$ by removing λ_i and λ_j. Define $\epsilon_i := (-1)^{j-i+\lambda_i} 2^{\frac{\operatorname{sgn}(\lambda)+1}{2}}$.

Let $I(\lambda, r) \subseteq \{1, \ldots, n\}$ be the set of indices such that one of the three cases occurs. Similar to the Murnaghan-Nakayama formula, $\hat{\chi}_\lambda(\sigma)$ can be computed recursively by *Morris' formula*

$$\hat{\chi}_\lambda(g) = \sum_{i \in I(\lambda,r)} \epsilon_i \hat{\chi}_{\lambda \backslash_i r}(h).$$

Note that these values are always rational since $\operatorname{sgn}(\mu) = 1$. The formula also applies artificially for $n < 4$ by setting $\hat{\chi}_{(1)}(1) := \hat{\chi}_{()}(1) := 1$. For $r = 1$, one can deduce analoga of the branching rules.

Example 9. Let $\lambda = (4, 2, 1)$, $\mu = (3, 3, 1)$ and $r = 3$. Then $I(\lambda, r) = \{2\}$ and $\hat{\chi}_\lambda(g) = -2\hat{\chi}_{(4)}(h)$. For $\mu = (3, 1)$ we can again choose $r = 3$ and obtain $\hat{\chi}_{(4)}(h) = \hat{\chi}_{(1)}(1) = 1$ and $\hat{\chi}_\lambda(g) = -2$. Alternatively, we can choose $r = 1$ from the beginning. This yields $I(\lambda, r) = \{1, 3\}$ and

$$\hat{\chi}_\lambda(g) = 2\hat{\chi}_{(3,2,1)}(h) + \hat{\chi}_{(4,2)}(h) = 2\big(\hat{\chi}_{(2,1)}(k) - \hat{\chi}_{(3)}(k)\big) - 2\hat{\chi}_{(2,1)}(k) = -2\hat{\chi}_{(3)}(k) = -2.$$

The spin characters $\tilde{\chi}_\lambda$ of the other covering group $\widetilde{S}_n$ can also be labeled by partitions λ with distinct parts. If $g \in \widetilde{S}_n$ has type λ and $\operatorname{sgn}(\lambda) = -1$, then $\tilde{\chi}_\lambda(g) = \sqrt{-1}\hat{\chi}_\lambda(g)$. On the remaining elements, $\tilde{\chi}_\lambda$ and $\hat{\chi}_\lambda$ coincide (e. g. $\tilde{\chi}_{(6)} = \operatorname{sgn} \hat{\chi}_{(3,2,1)}$).

The *shifted Young diagram* $\widehat{Y}_\lambda$ associated to λ is obtained by shifting the i-th row of the Young diagram $i - 1$ boxes to the right (so a staircase emerges on the left). The *bar lengths* of the i-row of $\widehat{Y}_\lambda$ contains the numbers

$$\{1, \ldots, \lambda_i\} \cup \{\lambda_i + \lambda_j : j > i\} \setminus \{\lambda_i - \lambda_j : j > i\}$$

in decreasing order (so $\lambda_i - \lambda_j$ is replaced by $\lambda_i + \lambda_j$). The (i, j)-th bar length is denoted by $|\overline{h}_{ij}|$ (despite shifting, the i-th row still starts with $(i, 1)$). The actual *bars* $\overline{h}_{ij}$ can be visualized as follows: If $i + j > l$, then $\overline{h}_{ij}$ consists of the last $\overline{h}_{ij}$ boxes in row i of $\widehat{Y}_\lambda$ (this is called an *unmixed bar*). If $i + j \leq l$, then $\overline{h}_{ij}$ consists of all boxes in rows i and $i + j$ of $\widehat{Y}_\lambda$ (a *mixed bar*).

Example 10. The shifted Young diagram and the bar lengths for $\lambda = (5, 4, 2, 1)$ are

9	7	6	5	2
	6	5	4	1
		2	3	
			1	

The mixed bars correspond to the blue boxes.

The analog to the hook formula is

$$\hat{\chi}_\lambda(1) = 2^{\lfloor \frac{n-l}{2} \rfloor} \frac{n!}{\prod |\overline{h}_{ij}|}.$$

We can remove a bar $\overline{h}_{ij}$ and rearrange the rows to obtain a new shifted Young diagram corresponding to a bar partition $\lambda \setminus \overline{h}_{ij}$.

Inclusion gives a one-to-one correspondence between the 2-blocks of S_n and $\widehat{S}_n$. If $B \subseteq \widehat{B}$ are such 2-blocks, then $l(B) = l(\widehat{B})$ and

$$k(\widehat{B}) = k(B) + p(w) + |\{\lambda \text{ partition of } w : \operatorname{sgn}(\lambda) = -(-1)^w\}|$$

where w is the weight of B.

Now let p be an odd prime. Every p-block of $\widehat{S}_n$ is a block of S_n or consists entirely of spin characters. In the latter case we call it a *spin block*. Bars of size p are called *p-bars*. Removing all p-bars from a bar partition λ successively yields the *$\overline{p}$-core* of λ. The number of removed p-bars is the *$\overline{p}$-weight* of λ (this equals the number of bar lengths divisible by p). Two spin characters lie in the same (spin) block $\widehat{B}$ if and only if they have the same $\overline{p}$-core (*Morris conjecture*). Moreover, $\operatorname{sgn}\widehat{B} = \widehat{B}$. We attach the $\overline{p}$-weight and $\overline{p}$-core also to $\widehat{B}$. For weights $w > 0$, the defect group of $\widehat{B}$ is a Sylow p-subgroup of S_{pw} (still assuming $p > 2$). However, for $w = 0$, the defect is 0 if $\operatorname{sgn}(\lambda) = 1$ and 1 otherwise (since $\operatorname{sgn}\lambda \in \widehat{B}$). In general, the *sign* of a spin block with $\overline{p}$-core μ is $\operatorname{sgn}(\mu)$. In contrast to S_n, the number $k(\widehat{B}) = k(w, \epsilon)$ of characters in $\widehat{B}$ does not only depend on the $\overline{p}$-weight w, but also on the sign ϵ. Let $q := (p-1)/2$ and

$$\sum_{k=0}^{\infty} \alpha(n, \epsilon)x^n := \frac{1}{2}\Big(\frac{P(x)^{q+1}}{P(x^2)} + \epsilon\frac{P(x^2)^{3q-3}}{P(x)^{q-1}P(x^4)^{q-1}}\Big).$$

Then $k(w, \epsilon) = \alpha(w, \epsilon) + 2\alpha(w, -\epsilon)$. For $p = 3$ and $w > 0$, the sign is irrelevant, i. e. $k(w, \epsilon) = k(w, -\epsilon)$. For $p = 5$, we have $k(w, (-1)^w) + p(w) = k(w, -(-1)^w)$. Brunat–Gramain have constructed perfect isometries between blocks of $2.S_n$ with the same weight and the same sign.

The Schur multiplier of A_n is

$$M(A_n) = \begin{cases} 2 & \text{if } n = 4, 5, 8, 9, \ldots \\ 6 & \text{if } n = 6, 7 \end{cases}$$

and in each case there exists a unique covering group $\widehat{A}_n$ (since A_n is perfect). For $n \neq 6, 7$ we have $\widehat{A}_n \cong \widehat{S}_n'$. For $n = 6, 7$ the covering groups are conveniently defined by GAP as `PerfectGroup`(2160) and `PerfectGroup`(15120) respectively. We may assume $n \geq 8$ for the remainder. For an odd bar partition λ the restriction of $\hat{\chi}_\lambda$ to $\widehat{A}_n$ is irreducible. If $\operatorname{sgn}(\lambda) = 1$, then the restriction is a sum of two irreducible characters $\hat{\xi}_\lambda$ and $\hat{\xi}_\lambda^{(1,2)}$ of $\widehat{A}_n$. Let $g \in \widehat{A}_n$ of type μ. Then

$$\hat{\xi}_\lambda(g) = \begin{cases} \frac{1}{2}\big(\hat{\chi}_\lambda(g) \pm \sqrt{-1}^{\frac{n-l}{2}}\sqrt{\lambda_1 \ldots \lambda_l}\big) & \text{if } \mu = \lambda, \\ \frac{1}{2}\hat{\chi}_\lambda(g) & \text{if } \mu \neq \lambda. \end{cases}$$

Every spin block $\widehat{B}$ of $\widehat{S}_n$ of weight $w > 0$ covers a unique block $\widehat{A}$ of $\widehat{A}_n$. Moreover, $k(\widehat{A}) = k(w, -\epsilon)$ where ϵ is the sign of $\widehat{B}$.

The perfect isometries constructed by Brunat–Gramain can be adapted to $2.A_n$. Moreover, Broué's conjecture for blocks of $2.S_n$ and $2.A_n$ was verified by Livesey.