Synthetic Geometry

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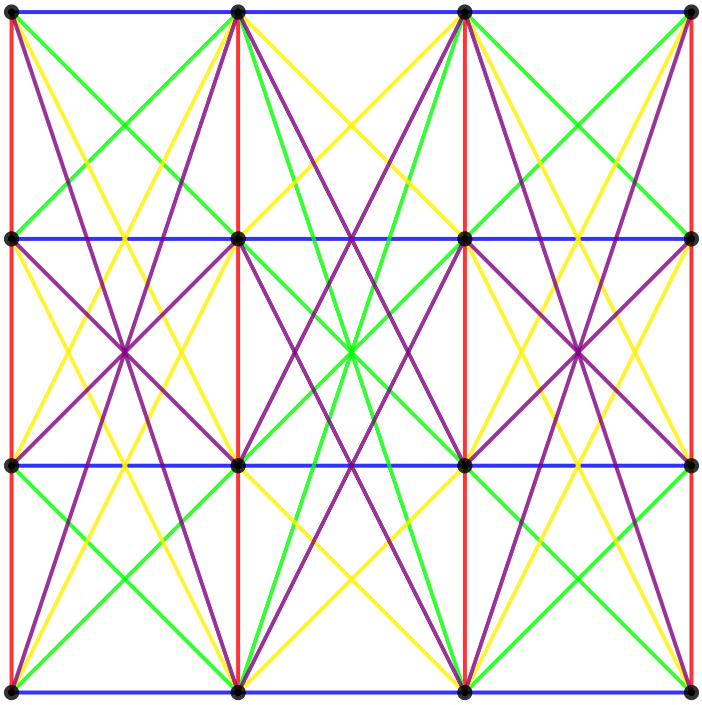

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Warning: This is an AI-translated version of my German lecture notes, performed by *Gemini 3 Flash Preview*. I have not checked whether Gemini introduced errors. Use with care!

Preface

Most geometry books are mainly dedicated to *projective* planes. These notes are an attempt to build the geometry of the *affine* plane axiomatically. The main source is [1], although unfortunately the connection between translation planes and Desarguesian planes is not fully clarified there. This gap is closed in [2]. In some places I use knowledge of algebra (division rings, Galois theory, Brauer group), which can be read in my lecture notes on algebra. Many typos have been corrected by Claude Sonnet 4.6.

References

[1] M. Koecher and A. Krieg, *Ebene Geometrie*, 3rd edition, Springer, Berlin, 2007

[2] P. Scherk and R. Lingenberg, *Rudiments of plane affine geometry*, University of Toronto Press, 1975

1 Affine Planes

Definition 1.1. Let $\mathcal{P}$ be a set and $\mathcal{G}$ a set of subsets of $\mathcal{P}$.

- The elements of $\mathcal{P}$ and $\mathcal{G}$ are called *points* and *lines*, respectively.
- We say $x \in \mathcal{P}$ *lies on* $G \in \mathcal{G}$ or G *passes through* x, if $x \in G$.
- Points on a line are called *collinear*.
- If x lies on G and H, then x is called an *intersection point* of G and H.
- If the lines G and H are equal or disjoint, they are called parallel. One also says G is a *parallel* of H and writes $G \parallel H$.
- Points $x, y, z \in \mathcal{P}$ are in *general position*, if they are not collinear.

The pair $(\mathcal{P}, \mathcal{G})$ is called an *affine plane*, if the following hold:

(A1) Every line contains at least two points.

(A2) Any two distinct points $x, y \in \mathcal{P}$ lie on exactly one line $G \in \mathcal{G}$. One writes $xy := G$.

(A3) (Parallel axiom) For every $x \in \mathcal{P}$ and $G \in \mathcal{G}$ there exists exactly one parallel of G passing through x.

(A4) There exist three points in general position.

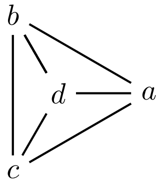
Remark 1.2.

(i) The parallel axiom goes back to Euclid's "Elements". For 2000 years, attempts were made to deduce it from the other axioms, until Gauss showed that this is impossible.

(ii) To avoid premature conclusions, one should not interpret the terms “point”, “line” and “plane” too intuitively. Hilbert wrote that they could be replaced by “table”, “chair” and “beer mug”.

Example 1.3.

(i) Let $(\mathcal{P},\mathcal{G})$ be an affine plane and $a,b,c\in\mathcal{P}$ in general position. According to the parallel axiom, there exists a line G through c that is parallel to ab. According to (A1), G possesses at least one further point d. Every affine plane thus possesses at least four points. If one chooses for $\mathcal{G}$ all (six) 2-element subsets of $\mathcal{P}=\{a,b,c,d\}$, then $(\mathcal{P},\mathcal{G})$ is an affine plane.



(ii) Let $\mathcal{P}:=\{(x,y)\in\mathbb{R}^2:y\ge 0\}$ be the upper half-plane and

$$\mathcal{G}:=\big\{(x,y)\in\mathcal{P}:y\ge 0\big\}\cup\big\{(x,y)\in\mathcal{P}:(x-\lambda)^2+y^2=\rho^2>0\big\}$$

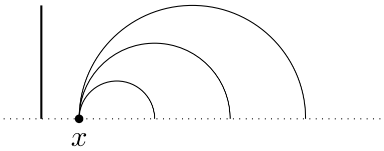
the set of all vertical rays and all semicircles with center on the x-axis. Every line possesses infinitely many points and there are certainly three points in general position. Now let (a,b) and (c,d) be distinct points in $\mathcal{P}$. In the case $a=c$, both points lie on exactly one vertical line and on no semicircle. In the case $a\ne c$, the approach

$$(a-\lambda)^2+b^2=\rho^2=(c-\lambda)^2+d^2$$

leads to the unique solution

$$\lambda:=\frac{a^2+b^2-c^2-d^2}{2(a-c)},\qquad\qquad \rho:=\sqrt{(a-\lambda)^2+b^2}=\sqrt{(c-\lambda)^2+d^2}.$$

Because of $a\ne c$, $(a-\lambda)^2>0$ or $(b-\lambda)^2>0$. In any case, $\rho>0$. Thus (A2) is satisfied. Otherwise, the parallel axiom is *not* satisfied, for there can even be infinitely many parallels to a line that all intersect in one point:



Remark 1.4. Large parts of linear algebra can be carried out more generally over *skew fields K* instead of fields (the multiplication in K does not have to be commutative). For example, the basis extension theorem as well as Steinitz’ exchange theorem hold in this generality (Algebra 2). In particular, the dimension for a K-vector space can be defined. For any two linearly independent vectors $v,w\in K^2$ and $v',w'\in K^2$, there exists exactly one $A\in\mathrm{GL}(2,K)$ with $Av=v'$ and $Aw=w'$. We will make use of this in the following.

Theorem 1.5. *Let K be a skew field and $\mathcal{P}=K^2$. For $a,v\in\mathcal{P}$ with $v\ne(0,0)$ let $G_{a,v}:=a+Kv$ be the line through a in* direction *Kv. Let $\mathcal{G}:=\{G_{a,v}:a,v\in\mathcal{P}:v\ne 0\}$. Then $\mathcal{E}(K):=(\mathcal{P},\mathcal{G})$ is an affine plane.*

Proof. Let $G_{a,v}, G_{b,w} \in \mathcal{G}$. If v and w are linearly dependent, then $G_{a,v}$ and $G_{b,w}$ are cosets of $Kv = Kw$ in $(\mathcal{P}, +)$. Consequently, $G_{a,v}$ and $G_{b,w}$ are then equal or disjoint. If v, w are linearly independent, they form a basis of K^2 and there exist unique $\lambda, \mu \in K$ with $\lambda v - \mu w = b - a$. Then $G_{a,v} \cap G_{b,w} = \{a + \lambda v = b + \mu w\}$. Conclusion: $G_{a,v} \parallel G_{b,w} \iff Kv = Kw$. Now for the axioms:

(A1) Obviously a and $a + v$ lie on $G_{a,v}$.

(A2) Distinct $x, y \in \mathcal{P}$ lie on $G_{x,y-x}$. If also $x, y \in H \in \mathcal{G}$ holds, then $H = G_{x,y-x}$ follows as above.

(A3) Let $x \in \mathcal{P}$ and $G := G_{a,v} \in \mathcal{G}$. Then $x \in H := G_{x,v} \parallel G$. If also $x \in H' := G_{b,w} \parallel G$ holds, one obtains $Kw = Kv$ and $H \parallel H'$. Because of $x \in H \cap H'$, it follows that $H = H'$.

(A4) Obviously $(0, 0)$, $(1, 0)$ and $(0, 1)$ are in general position. ☐

Remark 1.6. One calls $\mathcal{E}(K)$ the *coordinate plane* over K. For $K = \mathbb{F}_2$ one obtains the plane with four points from Example 1.3. The cover image shows $\mathcal{E}(\mathbb{F}_4)$, where lines with the same “slope” have the same color. The special case $\mathcal{E}(\mathbb{R})$ is called the *Euclidean* plane. In the following, let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ always be an affine plane.

Lemma 1.7. *For $G, H \in \mathcal{G}$ the following holds:*

(i) From $G \subseteq H$ follows $G = H$.

(ii) From $G \nparallel H$ follows $G \cap H = \{x\}$. One then writes $G \wedge H := x$.

Proof.

(i) According to (A1) there exist distinct $x, y \in G \subseteq H$. According to (A2) it is $G = xy = H$.

(ii) Since G and H are not parallel, there exists an intersection point $x \in G \cap H$. If $y \neq x$ is also an intersection point, the contradiction $G = xy = H$ follows. ☐

Lemma 1.8. *Parallelism is an equivalence relation on $\mathcal{G}$.*

Proof. Let $(\mathcal{P}, \mathcal{G})$ be an affine plane and $G, H, K \in \mathcal{G}$. Certainly $G \parallel G$ and $G \parallel H \Leftrightarrow H \parallel G$. Now let $G \parallel H \parallel K$ and wlog. G, H, K be pairwise distinct. Suppose G and K have an intersection point x. Then both G and K are the uniquely determined parallel of H through x. Thus $G = K$ and $G \parallel K$. ☐

Definition 1.9. The set of parallels of $G \in \mathcal{G}$ is called a *direction* and is written as $[G]$. Let the set of all directions be $[\mathcal{G}]$. The set of lines through $x \in \mathcal{P}$ is called a *pencil of lines* and is written as $[x]$. Let the set of all pencils of lines be $[\mathcal{P}]$.

Theorem 1.10. *Let $G, H \in \mathcal{G}$ be not parallel. Then $\varphi \colon [H] \to G$, $K \mapsto K \wedge G$ is a bijection. In particular, any two lines have the same cardinality.*

Proof. According to Lemma 1.7 and the parallel axiom, φ is well-defined and bijective. If also $G' \nparallel H$, one can replace G by G' and obtains $|G| = |[H]| = |G'|$. Now let $x, y \in H$ be distinct. Then there exists a $z \in \mathcal{P} \setminus H$ according to (A4). Now $H = xy$, $G := xz$ and $G' := yz$ are pairwise not parallel. It follows that $|H| = |[G]| = |G'| = |G|$. ☐

Definition 1.11. The cardinality of a line is called the *order* $\operatorname{ord}(\mathcal{E})$ of $\mathcal{E}$ (according to Theorem 1.10 this is well-defined).

Theorem 1.12. *For the order $n \in \mathbb{N} \cup \{\infty\}$ of $(\mathcal{P}, \mathcal{G})$, the following holds:*

(i) $|[x]| = |[G]| + 1 = n + 1$ *for all* $x \in \mathcal{P}$ *and* $G \in \mathcal{G}$.

(ii) $|\mathcal{P}| = n^2$ *and* $|\mathcal{G}| = n^2 + n$.

Proof.

(i) The equation $|[G]| = n$ already follows from Theorem 1.10. As usual, there exists $H \in \mathcal{G}$ with $x \notin H$. Every point $y \in H$ determines the line $xy \in [x]$. Additionally, there is exactly one parallel of H through x. This shows $|[x]| = n + 1$.

(ii) Since every direction contains exactly one line through x (parallel axiom), it holds that $|\mathcal{G}| = |[x]||[G]| = n^2 + n$ according to (i). Now $|\mathcal{P}|(n + 1) = |\mathcal{P}||[x]| = \sum_{G \in \mathcal{G}} |G| = |\mathcal{G}|n = n^2(n + 1)$ and $|\mathcal{P}| = n^2$. □

Remark 1.13. There are no affine planes with exactly 5 points, since 5 is not a square number. For every prime power $q \neq 1$, there exists a field with q elements. The coordinate plane then has order q. BRUCK and RYSER have proven that the order n of an affine plane is the sum of two square numbers if n is congruent to 1 or 2 modulo 4. This excludes $n = 6$. Using extensive computer resources, it has been shown that there are also no affine planes of order 10. It is conjectured that the order of a finite affine plane is always a prime power, even if not every plane is a coordinate plane.

2 Isomorphisms

Definition 2.1. Let $\mathcal{E} := (\mathcal{P}, \mathcal{G})$ and $\mathcal{E}' := (\mathcal{P}', \mathcal{G}')$ be affine planes. A bijection $\varphi \colon \mathcal{P} \to \mathcal{P}'$ is called an *isomorphism* if $\varphi(G) \in \mathcal{G}'$ for all $G \in \mathcal{G}$ (lines are mapped to lines). If applicable, $\mathcal{E}$ and $\mathcal{E}'$ are called isomorphic and one writes $\mathcal{E} \cong \mathcal{E}'$. In the case $\mathcal{E} = \mathcal{E}'$, φ is called an *automorphism* (or *collineation*) of $\mathcal{E}$. The automorphisms of $\mathcal{E}$ form the *automorphism group* $\operatorname{Aut}(\mathcal{E})$ with respect to composition of maps.

Remark 2.2. Let $\varphi \colon \mathcal{E} \to \mathcal{E}'$ be an isomorphism. For distinct $x, y \in \mathcal{P}$, it holds that $\varphi(xy) = \varphi(x)\varphi(y)$. In particular, φ also induces a bijection $\mathcal{G} \to \mathcal{G}'$. For $G, H \in \mathcal{G}$, it holds that $G \parallel H \Leftrightarrow \varphi(G) \parallel \varphi(H)$. Furthermore, $\operatorname{Aut}(\mathcal{E}) \to \operatorname{Aut}(\mathcal{E}')$, $\sigma \mapsto \varphi\sigma\varphi^{-1}$ is an isomorphism of groups.

Example 2.3.

(i) Every affine plane $(\mathcal{P}, \mathcal{G})$ of order 2 is isomorphic to $\mathcal{E}(\mathbb{F}_2)$, because $\mathcal{G}$ must consist of all 2-element subsets of $\mathcal{P}$. Every permutation $\mathcal{P} \to \mathcal{P}$ is an automorphism, i. e. $\operatorname{Aut}(\mathcal{E}(\mathbb{F}_2)) = \operatorname{Sym}(\mathcal{P}) \cong S_4$.

(ii) Let K be a skew field, $A \in \operatorname{GL}(2, K)$, $x \in K^2$ and $\alpha \in \operatorname{Aut}(K)$. For $v = (v_1, v_2) \in K^2$ let $\alpha(v) = (\alpha(v_1), \alpha(v_2))$. The map

$$\Gamma_{A,x,\alpha} \colon K^2 \to K^2, \qquad\qquad v \mapsto \alpha(v)A + x$$

is clearly injective. Since $\alpha^{-1}((v - x)A^{-1})$ is a preimage of v, $\Gamma_{A,x,\alpha}$ is bijective. For a line $G_{a,v}$ it holds that

$$\Gamma_{A,x,\alpha}(G_{a,v}) = \alpha(a + Kv)A + x = \alpha(a)A + K\alpha(v)A + x = G_{\alpha(a)A+x,\alpha(v)A}.$$

This shows $\Gamma_{A,x,\alpha} \in \operatorname{Aut}(\mathcal{E}(K))$. We show in Theorem 3.9 that every automorphism of $\mathcal{E}(K)$ has this form.

Definition 2.4. A permutation $\sigma\colon \mathcal{P} \to \mathcal{P}$ is called a *dilatation* of $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ if $\sigma(x)\sigma(y) \parallel xy$ holds for all $x, y \in \mathcal{P}$.

Theorem 2.5. *The dilatations form a normal subgroup* $\operatorname{Dil}(\mathcal{E})$ *of* $\operatorname{Aut}(\mathcal{E})$*.*

Proof. Clearly $\operatorname{Dil}(\mathcal{E})$ is a group. For $\sigma \in \operatorname{Dil}(\mathcal{E})$, $\varphi \in \operatorname{Aut}(\mathcal{E})$ and $x \neq y$ it holds that

$$\begin{aligned} &\sigma(\varphi^{-1}(x))\sigma(\varphi^{-1}(y)) \parallel \varphi^{-1}(x)\varphi^{-1}(y) \\ (\varphi\sigma\varphi^{-1})(x)(\varphi\sigma\varphi^{-1})(y) = \varphi(\sigma(\varphi^{-1}(x))&\sigma(\varphi^{-1}(y))) \parallel \varphi(\varphi^{-1}(x)\varphi^{-1}(y)) = xy. \end{aligned}$$

This shows $\varphi\sigma\varphi^{-1} \in \operatorname{Dil}(\mathcal{E})$.

We only need to show $\operatorname{Dil}(\mathcal{E}) \subseteq \operatorname{Aut}(\mathcal{E})$. For $\operatorname{ord}(\mathcal{E}) = 2$, we have $\operatorname{Dil}(\mathcal{E}) \subseteq \operatorname{Sym}(\mathcal{P}) = \operatorname{Aut}(\mathcal{E})$ by Example 2.3. Now let $\operatorname{ord}(\mathcal{E}) \geq 3$ and x, y, z be pairwise distinct points on $G \in \mathcal{G}$. For $\sigma \in \operatorname{Dil}(\mathcal{E})$ it holds that

$$\sigma(x)\sigma(z) \parallel xz = xy = yz \parallel \sigma(y)\sigma(z).$$

By Lemma 1.8, $\sigma(x)\sigma(z) \parallel \sigma(y)\sigma(z)$. Since both lines contain $\sigma(z)$, it follows that

$$\sigma(x)\sigma(z) = \sigma(y)\sigma(z) = \sigma(x)\sigma(y).$$

This shows $\sigma(G) \subseteq \sigma(x)\sigma(y)$. Replacing σ by σ^{-1}, one obtains $\sigma^{-1}(\sigma(x)\sigma(y)) \subseteq xy = G$. Thus $\operatorname{Dil}(\mathcal{E}) \subseteq \operatorname{Aut}(\mathcal{E})$. $\square$

Lemma 2.6. *Every dilatation is uniquely determined by the images of two distinct points. In particular,* id *is the only dilatation with two fixed points.*

Proof. Let $\sigma \in \operatorname{Dil}(\mathcal{E})$ and $x, y \in \mathcal{P}$ be distinct. First, let $z \in \mathcal{P} \setminus xy$. Let G be the parallel of xz through $\sigma(x)$ and let H be the parallel of yz through $\sigma(y)$. Because of $\sigma(x) \in G \parallel xz \parallel \sigma(x)\sigma(z)$, it follows that $\sigma(z) \in G$ and analogously $\sigma(z) \in H$. From $G \parallel xz \nparallel yz \parallel H$ it now follows that $\sigma(z) = G \wedge H$.

Now let $z \in xy \setminus \{x, y\}$. We choose $a \in \mathcal{P} \setminus xy$. According to the first part, $\sigma(a)$ is uniquely determined by $\sigma(x)$ and $\sigma(y)$. Because of $z \notin xa$, $\sigma(z)$ is uniquely determined by $\sigma(x)$ and $\sigma(a)$. The claim follows. $\square$

Example 2.7. For a field K, the maps $\Gamma_{\lambda 1_2, x, \mathrm{id}}$ with $\lambda \in K^\times$ and $x \in K^2$ are dilatations of $\mathcal{E}(K)$, because

$$\Gamma_{\lambda 1_2, x, \mathrm{id}}(G_{a,v}) = G_{\lambda a + x, \lambda v} \parallel G_{a,v}.$$

Conversely, let $\Gamma \in \operatorname{Dil}(\mathcal{E}(K))$. By multiplying Γ by $\Gamma_{1_2, -\Gamma(0), \mathrm{id}}$, one can assume $\Gamma(0) = 0$. Let $v := (1, 0)$. Because of $0\Gamma(v) = \Gamma(0)\Gamma(v) \parallel 0v = G_{0,v} = Kv$, there exists a $\lambda \in K^\times$ with $\Gamma(v) = \lambda v$. Now Γ and $\Gamma_{\lambda 1_2, 0, \mathrm{id}}$ coincide at the points 0 and v. From Lemma 2.6 it follows that $\Gamma = \Gamma_{\lambda 1_2, 0, \mathrm{id}}$. In total, it holds that

$$\operatorname{Dil}(\mathcal{E}(K)) = \{\Gamma_{\lambda 1_2, x, \mathrm{id}} : \lambda \in K^\times, x \in K^2\} \cong K^2 \rtimes K^\times.$$

Definition 2.8. One calls $G \in \mathcal{G}$ a *fixed line* of $\varphi \in \operatorname{Aut}(\mathcal{E})$ if $\varphi(G) = G$ holds.

Theorem 2.9. *For every dilatation* $\sigma \neq \mathrm{id}$*, exactly one of the following statements holds:*

(i) *The fixed lines of* σ *form a direction. Then* σ *has no fixed points.*

(ii) *The fixed lines of* σ *form a pencil of lines* $[x]$*, where* x *is the only fixed point of* σ*.*

Proof. Because $\sigma \neq \mathrm{id}$, there exists $a \in \mathcal{P}$ with $\sigma(a) \neq a$. From $\sigma(a)a \parallel \sigma^2(a)\sigma(a)$ it follows that $\sigma(\sigma(a)a) = \sigma(a)a$. Thus $G := \sigma(a)a$ is a fixed line of σ. Let us assume that σ has a fixed point $x \in \mathcal{P}$. According to Lemma 2.6, x is then the only fixed point of σ. Let $y \in G \setminus \{x\}$. Then $x = \sigma(x) \in \sigma(x)\sigma(y) \parallel xy$ and $\sigma(x)\sigma(y) = xy = y\sigma(y) = G$. Thus $G \in [x]$. Conversely, if $H \in [x]$, then $x = \sigma(x) \in \sigma(H) \parallel H$ and $\sigma(H) = H$.

Now we assume that σ has no fixed points. Every fixed line must then be parallel to G, since otherwise $G \wedge H$ would be a fixed point of σ. Conversely, let $H \in [G]$ and $y \in H$. Then $y\sigma(y)$ is a fixed line and therefore parallel to G and to H. Because $y \in y\sigma(y) \cap H$, it follows that $\sigma(y) \in y\sigma(y) = H$. This shows $\sigma(H) = H$. ☐

Remark 2.10. If the fixed lines of a dilatation σ form a direction $[G]$, then $[G]$ is also called the *direction* of σ.

Definition 2.11. A dilatation σ is called a *translation*, if σ has no fixed points or if $\sigma = \mathrm{id}$. Let $\mathrm{Tra}(\mathcal{E})$ be the set of translations of $\mathcal{E}$.

Lemma 2.12. *Every translation is uniquely determined by the image of one point.*

Proof. Let $\sigma \in \mathrm{Tra}(\mathcal{E})$ and $x \in \mathcal{P}$. In the case $\sigma(x) = x$, we have $\sigma = \mathrm{id}$. Otherwise $G := x\sigma(x)$ is a fixed line of σ. According to Lemma 2.6, it suffices to show that the image of another point $y \in \mathcal{P} \setminus G$ is uniquely determined. The fixed line $H := y\sigma(y)$ must be the parallel to G through y according to Theorem 2.9. Furthermore, $K := \sigma(x)\sigma(y)$ is the parallel to xy through $\sigma(x)$. Because $y \notin G$, we have $H \parallel G \nparallel xy \parallel K$. It follows that $\sigma(y) = H \wedge K$. ☐

Example 2.13. For every field K, the maps $\Gamma_{1_2,x,\mathrm{id}}$ with $x \in K^2$ are translations. According to Lemma 2.12, there can be no other translations.

Theorem 2.14. *For every translation plane,* $\mathrm{Tra}(\mathcal{E}) \trianglelefteq \mathrm{Aut}(\mathcal{E})$ *holds.*

Proof. Let $\sigma, \tau \in \mathrm{Tra}(\mathcal{E})$. Since σ and σ^{-1} have the same fixed points, $\sigma^{-1} \in \mathrm{Tra}(\mathcal{E})$ holds. If $\sigma\tau$ has a fixed point x, then $\tau(x) = \sigma^{-1}(x)$ and Lemma 2.12 shows $\tau = \sigma^{-1}$. It follows that $\sigma\tau = \mathrm{id} \in \mathrm{Tra}(\mathcal{E})$. Otherwise $\sigma\tau \in \mathrm{Dil}(\mathcal{E})$ has no fixed points and one obtains $\sigma\tau \in \mathrm{Tra}(\mathcal{E})$. Thus $\mathrm{Tra}(\mathcal{E}) \leq \mathrm{Aut}(\mathcal{E})$. For $\varphi \in \mathrm{Aut}(\mathcal{E})$ and $\tau \in \mathrm{Tra}(\mathcal{E})$, we have $\varphi\tau\varphi^{-1} \in \mathrm{Dil}(\mathcal{E})$ according to Theorem 2.5. If τ has no fixed points, then neither does $\varphi\tau\varphi^{-1}$. This shows $\varphi\tau\varphi^{-1} \in \mathrm{Tra}(\mathcal{E})$. ☐

3 Translation planes

Definition 3.1. An affine plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ is called a *translation plane*, if for all $x, y \in \mathcal{P}$ there exists (exactly) one translation τ with $\tau(x) = y$.

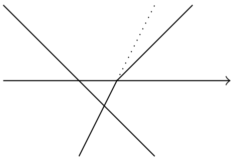
Example 3.2.

(i) For every field K, $\mathcal{E}(K)$ is a translation plane according to Example 2.13.

(ii) For $\alpha, \beta, \gamma \in \mathbb{R}$ with $(\alpha, \beta) \neq (0,0)$, we consider the following "lines" in $\mathcal{P} = \mathbb{R}^2$:

$$G_{\alpha,\beta,\gamma} := \begin{cases} \{(x,y) \in \mathbb{R}^2 : \alpha x + \beta y = \gamma\} & \text{if } \alpha\beta \geq 0 \\ \{(x,y) \in \mathbb{R}^2 : \alpha x + \beta y = \gamma, y \leq 0\} & \text{if } \alpha\beta < 0\,. \\ \quad \cup \{(x,y) \in \mathbb{R}^2 : \alpha x + 2\beta y = \gamma, y > 0\} & \end{cases}$$

(The lines with positive slope are bent at the x-axis.)

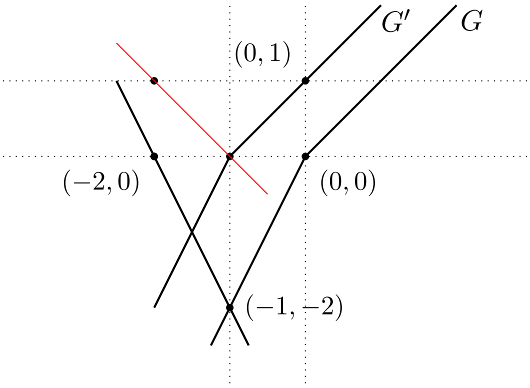


Let $\mathcal{G}$ be the set of these lines. We claim that $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ is an affine plane. Every line contains infinitely many points. Let $(x,y), (x',y') \in \mathcal{P}$ be distinct and wlog. $y \leq y'$. Then

$$M := \begin{cases} \begin{pmatrix} x & y & -1 \\ x' & y' & -1 \end{pmatrix} & \text{if } y' \leq 0 \\ \begin{pmatrix} x & y & -1 \\ x' & 2y' & -1 \end{pmatrix} & \text{if } y \leq 0 < y' \\ \begin{pmatrix} x & 2y & -1 \\ x' & 2y' & -1 \end{pmatrix} & \text{if } y > 0 \end{cases}$$

has rank 2. Thus the system of equations $(\alpha, \beta, \gamma)M^{\mathrm{t}} = 0$ has exactly one non-trivial solution (α, β, γ) up to scaling. Therefore, (x,y) and (x',y') lie on exactly one (bent) line. Two (bent) lines with different slopes always have an intersection point. Hence the parallel axiom also holds. The points $(0,0)$, $(1,0)$ and $(0,1)$ are in general position. Thus $\mathcal{E}$ is an affine plane, which is called the *Moulton plane*.

Suppose there exists a translation σ of $\mathcal{E}$ with $\sigma(0,0) = (0,1)$. The fixed lines of σ are the parallels to the y-axis, i. e. $\sigma(x,y) = (x,*)$ for all $(x,y) \in \mathcal{P}$. Furthermore, $\sigma(x,0) = (x,1)$. The bent line G with parameters $(\alpha,\beta,\gamma) = (1,-1,0)$ is mapped under σ to the parallel G' with parameters $(\alpha,\beta,\gamma) = (1,-1,-1)$. This shows $\sigma(-1,-2) = (-1,0)$. Now, however, the ordinary lines $(-2,0)(-1,-2)$ and $\sigma(-2,0)\sigma(-1,-2) = (-2,1)(-1,0)$ are not parallel. Therefore, $\mathcal{E}$ is *not* a translation plane.



Lemma 3.3. *For every translation plane $\mathcal{E}$, $\mathrm{Tra}(\mathcal{E})$ is abelian.*

Proof. Let $\sigma, \tau \in \mathrm{Tra}(\mathcal{E}) \setminus \{\mathrm{id}\}$ with directions $[G]$ and $[H]$. Then both G and H are fixed lines of $\sigma\tau\sigma^{-1}\tau^{-1}$. In the case $[G] \neq [H]$, it follows that $\sigma\tau = \tau\sigma$ from Theorem 2.9. Now let $[G] = [H]$, $x \in G$ and $y \in \mathcal{P} \setminus G$. Since $\mathcal{E}$ is a translation plane, there exists a translation ρ with $\rho(x) = y$. Obviously, G is then not a fixed line of ρ. From the first part, it follows that $\sigma\rho = \rho\sigma$ and $\tau\rho = \rho\tau$. Thus, G is also not a fixed line of $\tau\rho$. Again it follows

$$\sigma\tau = \sigma(\tau\rho)\rho^{-1} = \tau(\rho\sigma)\rho^{-1} = \tau\sigma\rho\rho^{-1} = \tau\sigma. \qquad \square$$

Definition 3.4. Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a translation plane and $\mathbf{o} \in \mathcal{P}$ an arbitrary point. For $x \in \mathcal{P}$, let $\tau_x \in \mathrm{Tra}(\mathcal{E})$ with $\tau_x(\mathbf{o}) = x$. We define $x + y := (\tau_x\tau_y)(\mathbf{o})$. According to Lemma 3.3, $x + y = \tau_x(y) = \tau_y(x)$ holds.

Example 3.5. The choice $\mathbf{o} = (0, 0)$ in the coordinate plane of a skew field K yields the addition of vectors in K^2.

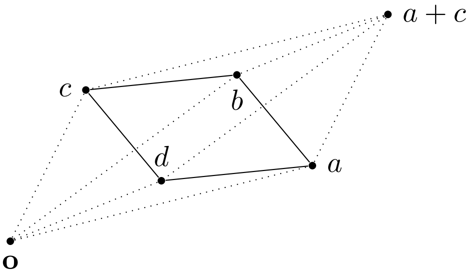
Lemma 3.6. *For every translation plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$, $(\mathcal{P}, +)$ is a group isomorphic to* $\mathrm{Tra}(\mathcal{E})$*. For $x \in \mathcal{P} \setminus \{\mathbf{o}\}$, $\mathbf{o}x$ is a subgroup of $(\mathcal{P}, +)$.*

Proof. The map $\tau \colon \mathcal{P} \to \mathrm{Tra}(\mathcal{E})$ is a bijection according to Lemma 2.12. Because of $\tau_{x+y}(\mathbf{o}) = x + y = (\tau_x\tau_y)(\mathbf{o})$ for $x, y \in \mathcal{P}$, $(\mathcal{P}, +)$ is a group and τ is an isomorphism of groups. For $x \neq \mathbf{o}$, $\mathbf{o}x$ is a fixed line of τ_x. This shows $x + y = \tau_x(y) \in \mathbf{o}x$ for $y \in \mathbf{o}x$. Thus, $\mathbf{o}x$ is a subgroup of $(\mathcal{P}, +)$. $\square$

Theorem 3.7 (Parallelogram Theorem)**.** *For pairwise distinct non-collinear points a, b, c, d of a translation plane, the following are equivalent:*

(1) $a + c = b + d$.

(2) $ab \parallel cd$, $ad \parallel bc$.



Proof. (1)⇒(2):

$$\begin{aligned} &ab \parallel \tau_{-a}(a)\tau_{-a}(b) = \mathbf{o}(b - a) = \mathbf{o}(c - d) \parallel dc = cd, \\ &ad \parallel \mathbf{o}(d - a) = \mathbf{o}(c - b) = bc. \end{aligned}$$

(2)⇒(1): Let $x \in \mathcal{P}$ with $b - a + x = \tau_x(b - a) = c - d$. Then it also holds that $\tau_x(d - a) = d - a + x = c - b$. Because of

$$\begin{aligned} &\mathbf{o}(b - a) \parallel ab \parallel dc \parallel \mathbf{o}(c - d), \\ &\mathbf{o}(d - a) \parallel ad \parallel bc \parallel \mathbf{o}(c - b), \end{aligned}$$

$\mathbf{o}(b - a) = \mathbf{o}(c - d)$ and $\mathbf{o}(d - a) = \mathbf{o}(c - b)$ are fixed lines of τ_x. In the case $ab \parallel ad$, a, b, c, d would be collinear. Thus $\mathbf{o} = \mathbf{o}(b - a) \wedge \mathbf{o}(d - a)$ is a fixed point of τ_x. This shows $\tau_x = \mathrm{id}$ and $x = \mathbf{o}$. $\square$

Lemma 3.8. *Every automorphism φ of a translation plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ with $\varphi(\mathbf{o}) = \mathbf{o}$ is also an automorphism of $\mathcal{P}$ with respect to $+$.*

Proof. Let first $\mathbf{o}, x, y \in \mathcal{P}$ be in general position. Because of $\mathbf{o}x \parallel y(x+y)$, it follows that $\mathbf{o}\varphi(x) \parallel \varphi(y)\varphi(x+y)$ and analogously $\mathbf{o}\varphi(y) \parallel \varphi(x)\varphi(x+y)$. On the other hand, $\mathbf{o}\varphi(x) \parallel \varphi(y)(\varphi(x)+\varphi(y))$ and $\mathbf{o}\varphi(y) \parallel \varphi(x)(\varphi(x)+\varphi(y))$. This shows $\varphi(y)\varphi(x+y) = \varphi(y)(\varphi(x)+\varphi(y))$ and $\varphi(x)\varphi(x+y) = \varphi(x)(\varphi(x)+\varphi(y))$. The unique intersection point of these lines is $\varphi(x+y) = \varphi(x)+\varphi(y)$.

Now let $\mathbf{o}, x, y$ be collinear and $z \in \mathcal{P} \setminus xy$. From $x(x+y) \parallel \mathbf{o}y = \mathbf{o}x$ it follows that $x(x+y) = \mathbf{o}x = \mathbf{o}y = \mathbf{o}(x+y)$. Therefore $\mathbf{o}, x+y, z$ and $\mathbf{o}, x, y+z$ as well as $\mathbf{o}, y, z$ are in general position. From the first part it follows that

$$\varphi(x+y)+\varphi(z) = \varphi(x+y+z) = \varphi(x)+\varphi(y+z) = \varphi(x)+\varphi(y)+\varphi(z)$$

and $\varphi(x+y) = \varphi(x)+\varphi(y)$. ☐

Theorem 3.9. *For every skew field K it holds that*

$$\operatorname{Aut}(\mathcal{E}(K)) = \{\Gamma_{A,x,\alpha} : A \in \mathrm{GL}(2,K), x \in K^2, \alpha \in \operatorname{Aut}(K)\} =: \mathrm{A\Gamma L}(2,K).$$

Proof. According to Example 2.3, $\mathrm{A\Gamma L}(2,K) \subseteq \operatorname{Aut}(\mathcal{E}(K))$. Conversely, let $\Gamma \in \operatorname{Aut}(\mathcal{E}(K))$ be given. After multiplication by $\Gamma_{1_2,-\Gamma(0),\mathrm{id}}$, we can assume $\Gamma(0) = 0$. Let $e_1 := (1,0)$ and $e_2 := (0,1)$. Because $0e_1 \nparallel 0e_2$, it follows that $0\Gamma(e_1) \nparallel 0\Gamma(e_2)$. In particular, $\Gamma(e_1)$ and $\Gamma(e_2)$ form a basis of K^2. Let $A \in \mathrm{GL}(2,K)$ with $\Gamma(e_i)A = e_i$ for $i = 1,2$. By replacing Γ with $\Gamma_{A,0,\mathrm{id}}\Gamma$, we can assume $\Gamma(e_i) = e_i$ for $i = 1,2$. Then $\Gamma(Ke_i) = Ke_i$. Let $\alpha_i \colon K \to K$ with $\Gamma(x,0) = (\alpha_1(x),0)$ and $\Gamma(0,y) = (0,\alpha_2(y))$ for all $x,y \in K$. According to Lemma 3.8, Γ is an automorphism on $(K^2,+)$. This shows $\Gamma(1,1) = \Gamma(e_1)+\Gamma(e_2) = (1,1)$ and $\Gamma(K(1,1)) = K(1,1)$. From $(\alpha_1(x),\alpha_2(x)) = \Gamma(x,x) \in K(1,1)$ it follows that $\alpha_1 = \alpha_2 =: \alpha$. Obviously, α is an automorphism of $(K,+)$. For $x,y \in K$ we have

$$(\alpha(xy),\alpha(x)) = \Gamma(xy,x) = \Gamma(x(y,1)) \in K\Gamma(y,1) = K(\alpha(y),1)$$

and one obtains $\alpha(xy) = \alpha(x)\alpha(y)$. Thus $\alpha \in \operatorname{Aut}(K)$ and $\Gamma_{1_2,0,\alpha} \in \mathrm{A\Gamma L}(2,K)$. ☐

Remark 3.10.

(i) For $K \in \{\mathbb{F}_p, \mathbb{Q}, \mathbb{R}\}$, $\operatorname{Aut}(K) = 1$ and $\operatorname{Aut}(\mathcal{E}(K)) = \mathrm{AGL}(2,K) \cong K^2 \rtimes \mathrm{GL}(2,K)$.

(ii) Not every automorphism of $(\mathcal{P},+)$ lies in $\operatorname{Aut}(\mathcal{E})$. For $K = \mathbb{F}_4$, for example,

$$\begin{aligned}|\operatorname{Aut}(K^2,+)| &= |\operatorname{Aut}(C_2^4)| = |\operatorname{GL}(4,2)| = 15 \cdot 14 \cdot 12 \cdot 8\\ &> 16 \cdot 15 \cdot 12 \cdot 2 = |K^2||\operatorname{GL}(2,K)||\operatorname{Aut}(K)| = |\mathrm{A\Gamma L}(2,K)|.\end{aligned}$$

Definition 3.11. Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a translation plane. An endomorphism φ of $(\mathcal{P},+)$ is called a *multiplier* of $\mathcal{E}$ if $\varphi(x) \in \mathbf{o}x$ for all $x \in \mathcal{P} \setminus \{\mathbf{o}\}$. Let $K(\mathcal{E})$ be the set of multipliers of $\mathcal{E}$.

Example 3.12. The *trivial* multiplier $x \mapsto \mathbf{o}$ ($x \in \mathcal{P}$) is denoted by $\mathbf{o}$. The identity on $\mathcal{P}$ is also a multiplier. In the coordinate plane $\mathcal{E}$ of a field K, $K(\mathcal{E}) = \{\Gamma_{\lambda,0,\mathrm{id}} : \lambda \in K\} \cong K$.

Lemma 3.13. *For every translation plane $\mathcal{E}$, $K(\mathcal{E}) \setminus \{\mathbf{o}\}$ consists exactly of the dilatations φ with $\varphi(\mathbf{o}) = \mathbf{o}$.*

Proof. Let $\varphi \in \mathrm{Dil}(\mathcal{E})$ with $\varphi(\mathbf{o}) = \mathbf{o}$. According to Lemma 3.8, φ is an endomorphism with $\mathbf{o}x \parallel \mathbf{o}\varphi(x)$ for all $x \in \mathcal{P}\setminus\{\mathbf{o}\}$. This shows $\varphi \in K(\mathcal{E})$. Now conversely, let $\varphi \in K(\mathcal{E}) \setminus \{\mathbf{o}\}$ and $x \in \mathcal{P}$ with $\varphi(x) = \mathbf{o}$. Let us assume $x \neq \mathbf{o}$. For $y \in \mathcal{P} \setminus \mathbf{o}x$, it holds that $\varphi(y) = \varphi(y-x) \in \mathbf{o}y \cap \mathbf{o}(y-x) = \mathbf{o}$. For $z \in \mathcal{P} \setminus \mathbf{o}y$, one obtains by the same argument $\varphi(z) = \mathbf{o}$. But then $\varphi = \mathbf{o}$. This contradiction shows $\mathrm{Ker}(\varphi) = \{\mathbf{o}\}$, i. e. φ is injective. For $x, y \in \mathcal{P}$ it holds that

$$\varphi(x)\varphi(y) \parallel \mathbf{o}(\varphi(y) - \varphi(x)) = \mathbf{o}\varphi(y-x) = \mathbf{o}(y-x) \parallel xy. \tag{3.1}$$

It remains to show the surjectivity of φ. Let $x \in \mathcal{P} \setminus \{\mathbf{o}\}$ and $y \in \mathcal{P} \setminus \mathbf{o}x$. Let $G := x\varphi(y)$ and let H be the parallel to G through y. In the case $G \parallel \mathbf{o}x$, it would be $G = \mathbf{o}x$ and $\varphi(y) \in \mathbf{o}x \cap \mathbf{o}y = \mathbf{o}$. Thus there exists $z := H \cap \mathbf{o}x$. Because of $y \notin \mathbf{o}x$, $z \neq y$ and $H = yz$. According to (3.1), $G = \varphi(y)\varphi(z)$. From $z \in \mathbf{o}x$ it finally follows $\varphi(z) = \mathbf{o}x \wedge G = x$. ☐

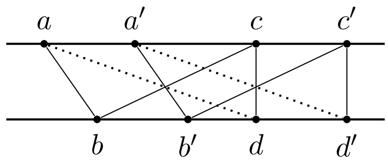
Theorem 3.14. *For every translation plane, $K(\mathcal{E})$ is a skew field.*

Proof. Since $(\mathcal{P}, +)$ is abelian, $\mathrm{End}(\mathcal{P})$ is a ring. By definition and Lemma 3.13, $K(\mathcal{E})$ is closed under addition and multiplication. The multipliers different from $\mathbf{o}$ are dilatations and therefore invertible. The claim follows. ☐

Remark 3.15. Note that the drawings in the following theorems always represent only a special case and cannot serve as a proof.

Theorem 3.16 (Little Shear Theorem)**.** *Let G and H be distinct parallels of a translation plane $\mathcal{E}$. For $a, a', c, c' \in G$ and $b, b', d, d' \in H$ it holds that*

$$ab \parallel a'b', \; bc \parallel b'c', \; cd \parallel c'd' \Longrightarrow ad \parallel a'd'.$$



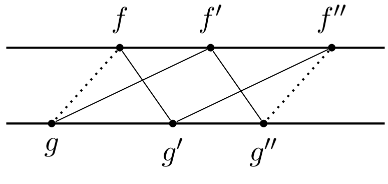
Proof. In the case $a = a'$, we have $b = ab \wedge H = a'b' \wedge H = b'$ and analogously $c = c'$, $d = d'$. We can therefore assume $a \neq a'$, $b \neq b'$ and $c \neq c'$. From the parallelogram theorem it follows that

$$a + (b' + c') + d' = a' + (b + c') + d' = a' + b' + (c + d') = a' + b' + c' + d,$$

thus $a + d' = a' + d$ and $ad \parallel a'd'$. ☐

Theorem 3.17 (Little Theorem of Pappus)**.** *Let F and G be distinct parallel lines of a translation plane $\mathcal{E}$. For $f, f', f'' \in F$ and $g, g', g'' \in G$ it holds that*

$$fg' \parallel f'g'', \; f'g \parallel f''g' \Longrightarrow fg \parallel f''g''.$$



Proof. In the case $f = f'$, we have $g' = fg' \wedge G = f'g'' \wedge G = g''$ and $fg = f'g \parallel f''g' = f''g''$. So let $f \neq f'$ and analogously $f' \neq f''$. From the parallelogram theorem it follows that $f+g'' = f'+g' = g+f''$ and $fg \parallel f''g''$. $\square$

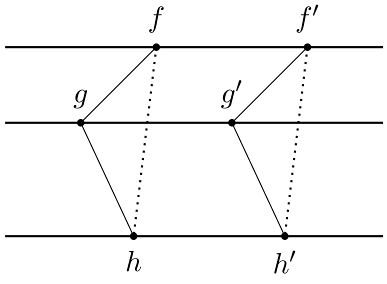
Remark 3.18. The following theorem gives a purely geometric characterization of translation planes.

Theorem 3.19 (Little Theorem of DESARGUES)**.** *For every affine plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ the following are equivalent:*

(1) $\mathcal{E}$ is a translation plane.

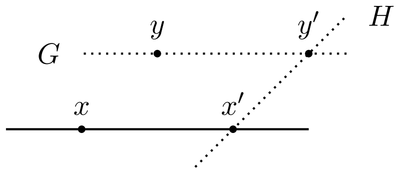
(2) For pairwise distinct parallel lines $F, G, H \in \mathcal{G}$, $f, f' \in F$, $g, g' \in G$ and $h, h' \in H$ it holds that

$$fg \parallel f'g',\ gh \parallel g'h' \implies fh \parallel f'h'.$$



Proof. (1)⇒(2): In the case $f = f'$, we have $g = fg \wedge G = f'g' \wedge G = g'$ and analogously $h = h'$. We can therefore assume $f \neq f'$ and $g \neq g'$. Then f, f', g, g' are not collinear. From the parallelogram theorem, it follows that $(f + g') + h' = f' + g + h' = f' + g' + h$, $f + h' = f' + h$ and $fh \parallel f'h'$.

(2)⇒(1): If $\mathcal{E}$ has order 2, then $\mathcal{E} = \mathcal{E}(\mathbb{F}_2)$ according to Example 2.3 and the assertion follows from Example 3.2. We can therefore assume $\operatorname{ord}\mathcal{E} \geq 3$. Let $x, x' \in \mathcal{P}$. We are looking for $\tau \in \operatorname{Tra}(\mathcal{E})$ with $\tau(x) = x'$. For this, we can assume $x \neq x'$. Let $y \in \mathcal{P} \setminus xx'$ and let G be the parallel to xx' through y. Let H be the parallel to xy through x'. Because $xx' \nparallel xy$, there exists $y' := G \wedge H$. Because $G \cap xx' = \varnothing$, we have $y' \notin xx'$. We define $\sigma_{x,x'} \colon \mathcal{P} \setminus xx' \to \mathcal{P} \setminus xx'$, $y \mapsto y'$. Because $H \cap xy = \varnothing$, we have $y' \neq y$ and it follows that $yy' = G \parallel xx'$ as well as $x'y' = H \parallel xy$. Therefore, the map $\sigma_{y,y'}$ with $\sigma_{y,y'}(x) = x'$ also exists.



Because $\operatorname{ord}\mathcal{E} \geq 3$, there exists $z \in \mathcal{P} \setminus (xx' \cup yy')$. For $z' := \sigma_{x,x'}(z)$, we have $xx' \parallel zz'$ and $xz \parallel x'z'$. From (2), it follows that $yz \parallel y'z'$. Thus, we also have $\sigma_{y,y'}(z) = z'$, i. e. $\sigma_{x,x'}$ and $\sigma_{y,y'}$ coincide outside of $xx' \cup yy'$. For reasons of symmetry, $\sigma_{y,y'}$ and $\sigma_{z,z'}$ coincide outside of $yy' \cup zz'$. Therefore, the map

$$\tau \colon \mathcal{P} \to \mathcal{P}, \qquad u \mapsto \begin{cases} \sigma_{x,x'}(u) & \text{if } u \notin xx', \\ \sigma_{y,y'}(u) & \text{if } u \in xx'. \end{cases}$$

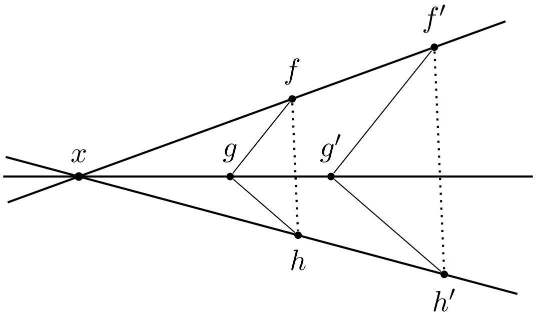
does not depend on the choice of y. Since $\tau(x) = x'$, it remains to show that τ is a translation. Let $u, v \in \mathcal{P}$ be distinct. One of the (disjoint) parallels xx', yy' or zz' contains neither u nor v. Wlog. let $xx' \cap \{u, v\} = \varnothing$. In the case $uv \parallel xx'$, we have $u\tau(u) = uu' \parallel xx' \parallel vv' = v\tau(v)$ and

$uv = u\tau(u) = v\tau(v) = \tau(u)\tau(v)$. Now let $uv \nparallel xx'$. Then one can choose $y = u$ and $z = v$ in the definition of τ. It follows that $\tau(u)\tau(v) = y'z' \parallel yz = uv$. Thus τ is a dilatation. By construction, τ has no fixed points, i. e. τ is a translation. □

4 Desarguesian planes

Definition 4.1. An affine plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ is called a *Desarguesian plane*, if the following property holds: Let $F, G, H \in \mathcal{G}$ with $x := F \wedge G = F \wedge H = G \wedge H$. For $f, f' \in F \setminus \{x\}$, $g, g' \in G \setminus \{x\}$ and $h, h' \in H \setminus \{x\}$ it holds that

$$fg \parallel f'g',\ gh \parallel g'h' \implies fh \parallel f'h'.$$

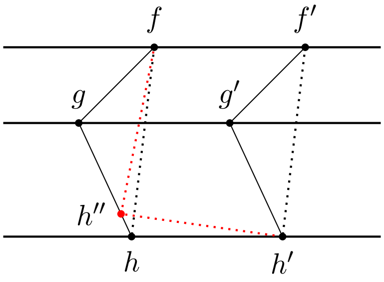


Theorem 4.2 (Converse of the Desargues property)**.** *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane and $f, g, h \in \mathcal{P}$ in general position. Let $f', g', h' \in \mathcal{P}$ with $fg \parallel f'g'$, $gh \parallel g'h'$, $fh \parallel f'h'$. Then the lines ff', gg', hh' are either parallel or they intersect in a point.*

Proof. Let wlog. $x := ff' \wedge gg'$ and $ff' \neq hh' \neq gg'$. Because of $fh \parallel f'h'$, then $h' \neq x$. In the case $fh \parallel xh'$, it would be $xh' = f'h'$ and $f' = f'h' \wedge ff' = x \in gg'$. So let $h'' := fh \wedge xh'$. In the case $h'' = f$, it would be $f \in xh' \wedge ff' = x$. Because of $fh = fh''$, we can apply the Desargues property to f, g, h'' and f', g', h'. It follows that $gh'' \parallel g'h' \parallel gh$, thus $h'' \in gh \cap fh = h$. This shows $x \in h'h'' = hh'$. □

Theorem 4.3. *Every Desarguesian plane is a translation plane.*

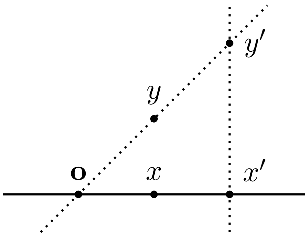
Proof. We use the minor theorem of Desargues on the Desarguesian plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$. Let $F, G, H \in \mathcal{G}$ be pairwise distinct parallel lines and $f, f' \in F$, $g, g' \in G$, $h, h' \in H$ with $fg \parallel f'g'$, $gh \parallel g'h'$. In the case $f'h' \parallel gh \parallel g'h'$, we have $fg \parallel f'g' = g'h' \parallel gh$ and $fh = gh \parallel g'h' = f'h'$. So let $f'h' \nparallel gh$. The parallel to $f'h'$ through f then intersects gh in a point h''. It holds that $h'' \neq h'$, because otherwise $h' \in F \cap H = \varnothing$. Because of $f'h' \parallel fh''$ and $gh'' = gh \parallel g'h'$, we can apply Theorem 4.2 to the points f, g, h'' and f', g', h'. From $F \parallel G$ it follows that $h'h'' \parallel F \parallel H$. This shows $h'' = H \wedge gh = h$ and $fh = fh'' \parallel f'h'$.



□

Lemma 4.4. *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane and $x \in \mathcal{P} \setminus \{\mathbf{o}\}$. For all $x' \in \mathbf{o}x$ there exists a multiplier α with $\alpha(x) = x'$.*

Proof. The proof is similar to the small theorem of Desargues. If $\mathcal{E}$ has order 2, then $x' = x$ and we can choose $\alpha = \mathrm{id}$. So let $\operatorname{ord}\mathcal{E} \geq 3$. Let $y \in \mathcal{P} \setminus \{\mathbf{o}x\}$ and $G \in \mathcal{G}$ be the parallel of xy through x'. Then $y' := \mathbf{o}y \wedge G$ exists and we define $\alpha_{x,x'} \colon \mathcal{P} \setminus \mathbf{o}x \to \mathcal{P} \setminus \mathbf{o}x$, $y \mapsto y'$. For a fixed $y \in \mathcal{P} \setminus \mathbf{o}x$, there exists $\alpha_{y,y'} \colon \mathcal{P} \setminus \mathbf{o}y \to \mathcal{P} \setminus \mathbf{o}y$ with $\alpha_{y,y'}(x) = x'$.



Because $\operatorname{ord}\mathcal{E} \geq 3$, there exists $z \in \mathcal{P} \setminus (\mathbf{o}x \cup \mathbf{o}y)$. For $z' := \alpha_{x,x'}(z)$, it holds that $xz \parallel x'z'$. The Desargues property shows $yz \parallel y'z'$ and $\alpha_{y,y'}(z) = z'$. Therefore, $\alpha_{x,x'}$ and $\alpha_{y,y'}$ coincide outside of $\mathbf{o}x \cup \mathbf{o}y$. For reasons of symmetry, $\alpha_{y,y'}$ and $\alpha_{z,z'}$ coincide outside of $\mathbf{o}y \cup \mathbf{o}z$. Therefore,

$$\alpha \colon \mathcal{P} \to \mathcal{P}, u \mapsto \begin{cases} \alpha_{x,x'}(u) & \text{if } u \notin \mathbf{o}x, \\ \alpha_{y,y'}(u) & \text{if } u \in \mathbf{o}x \setminus \{\mathbf{o}\}, \\ \mathbf{o} & \text{if } u = \mathbf{o} \end{cases}$$

does not depend on the choice of y. Because $\alpha(x) = x'$ and $\alpha(\mathbf{o}) = \mathbf{o}$, it suffices to show that α is a dilatation. Let $u, v \in \mathcal{P}$ be distinct. In the case $\mathbf{o} \in uv$, it holds that $\alpha(uv) = uv$. Otherwise, let wlog. $u, v \notin \mathbf{o}y$. As above, $yu \parallel y'u'$, $yv \parallel y'v'$ holds and it follows that $uv \parallel u'v' = \alpha(uv)$. □

Lemma 4.5. *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane with distinct points $x, y \in \mathcal{P}$. Then $xy = \{x + \alpha(y - x) : \alpha \in K(\mathcal{E})\}$.*

Proof. For the translation τ_x, it holds that

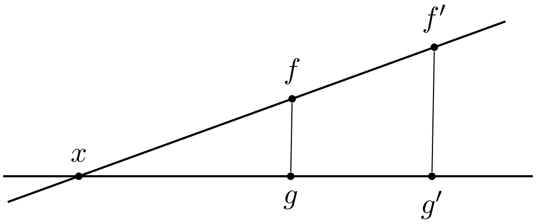
$$xy = \tau(\mathbf{o}(y - x)) \overset{4.4}{=} \tau(\{\alpha(y - x) : \alpha \in K(\mathcal{E})\}) = \{x + \alpha(y - x) : \alpha \in K(\mathcal{E})\}.$$ □

Theorem 4.6 (Intercept Theorem)**.** *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane and $F, G \in \mathcal{G}$ with $x = F \wedge G$. Let $f, f' \in F \setminus \{x\}$ and $g, g' \in G \setminus \{x\}$ be pairwise distinct. Then the following statements are equivalent:*

(1) $fg \parallel f'g'$.

(2) There exists $\gamma \in K(\mathcal{E})$ *with* $f' - x = \gamma(f - x)$ *and* $g' - x = \gamma(g - x)$.

(3) There exists $\gamma \in K(\mathcal{E})$ *with* $\gamma(g - f) = g' - f'$.



Proof. After translation we can assume $x = \mathbf{o}$. According to Lemma 4.4 there exist $\alpha, \beta \in K(\mathcal{E})$ with $\alpha(f) = f'$ and $\beta(g) = g'$. If $fg \parallel f'g'$ holds, then $f'\alpha(g) = \alpha(f)\alpha(g) \parallel fg \parallel f'g' = f'\beta(g)$ and $\alpha(g) = f'\alpha(g) \wedge \mathbf{o}g = f'\beta(g) \wedge \mathbf{o}g = \beta(g)$. From Lemma 2.6 it follows that $\alpha = \beta =: \gamma$.

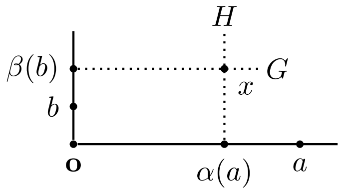
Now let $f' = \gamma(f)$ and $g' = \gamma(g)$ for some $\gamma \in K(\mathcal{E})$. Then $\gamma(g - f) = g' - f'$. Finally, let $\gamma(g - f) = g' - f'$. According to Lemma 4.5 it then holds that

$$f'g' = \{f' + \delta(g' - f') : \alpha \in K(\mathcal{E})\} = \{f' + \delta(g - f) : \delta \in K(\mathcal{E})\} = \tau_{f'-f}(fg) \parallel fg. \qquad \square$$

Lemma 4.7. *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane. Let $\mathbf{o}, a, b \in \mathcal{P}$ be in general position. For every $x \in \mathcal{P}$ there exist uniquely determined multipliers α, β with $x = \alpha(a) + \beta(b)$.*

Proof. For the existence of α and β, we can assume $x \notin \mathbf{o}a \cup \mathbf{o}b$. Let G and H be the parallel of $\mathbf{o}a$ and $\mathbf{o}b$ through x, respectively. Then there exist $\alpha, \beta \in K(\mathcal{E})$ with $\alpha(a) = G \wedge \mathbf{o}b$ and $\beta(b) = H \wedge \mathbf{o}a$. The translation $\tau_{\alpha(a)}$ maps $\mathbf{o}b = \mathbf{o}\beta(b)$ onto the parallel $\alpha(a)(\alpha(a) + \beta(b)) = G$. Analogously, $\tau_{\beta(b)}$ maps the line $\mathbf{o}a = \mathbf{o}\alpha(a)$ onto $\beta(b)(\beta(b) + \alpha(a)) = H$. Therefore,

$$\alpha(a) + \beta(b) = G \wedge H = x.$$



Now let also $\alpha', \beta' \in K(\mathcal{E})$ with $x = \alpha'(a) + \beta'(b)$. Then $(\alpha - \alpha')(a) + (\beta - \beta')(b) = \mathbf{o}$. We can therefore assume $x = \mathbf{o}$ and must show $\alpha = \beta = \mathbf{o}$. According to Theorem 3.14, it suffices to show $\alpha(a) = 0$. In the case $\alpha(a) \neq 0$, it would follow that $\mathbf{o}a = \mathbf{o}\alpha(a) = \mathbf{o}(-\beta(b)) = \mathbf{o}b$, in contradiction to the choice of a and b. $\square$

Theorem 4.8 (DESARGUES)**.** *A translation plane $\mathcal{E}$ is a Desarguesian plane if and only if it is isomorphic to the coordinate plane of the skew field $K(\mathcal{E})$.*

Proof. Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane and $K := K(\mathcal{E})$. Let $\mathbf{o}, a, b \in \mathcal{P}$ be in general position. According to Lemma 4.7, there exists a bijection

$$\Gamma\colon \mathcal{P} \to K^2, \qquad x \mapsto (\alpha_x, \beta_x)$$

with $x = \alpha_x(a) + \beta_x(b)$. Obviously, Γ is a homomorphism with respect to $+$. Let xy be any line in $\mathcal{E}$. According to Lemma 4.5, it holds that

$$\Gamma(xy) = \Gamma(x + K(y - x)) = \Gamma(x) + \Gamma(K(y - x)) = \Gamma(x) + K\Gamma(y - x).$$

This shows that Γ is an isomorphism of affine planes.

Conversely, let $\mathcal{E} = \mathcal{E}(K)$ with $K = K(\mathcal{E})$. In the Desargues property, we may assume $x = (0,0)$ after translation. Then there exist $\alpha, \beta, \gamma \in K$ with $f' = \alpha(f)$, $g' = \beta(g)$ and $h' = \gamma(h)$. Because $fg \parallel f'g'$, $f' - g' = \alpha(f) - \beta(g)$ and $f - g$ are linearly dependent. This yields $\alpha = \beta$. Analogously, $\beta = \gamma$. Therefore, $f' - h'$ and $f - h$ are also linearly dependent, i.e., $fh \parallel f'h'$. Consequently, $\mathcal{E}$ is a Desarguesian plane. ☐

Corollary 4.9. *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane and $f, f', g, g', h, h' \in \mathcal{P}$. If f, g, h and f', g', h' are each in general position, then there exists an $\alpha \in \operatorname{Aut}(\mathcal{E})$ with $\alpha(f) = f'$, $\alpha(g) = g'$ and $\alpha(h) = h'$.*

Proof. Wlog. let $\mathcal{E} = \mathcal{E}(K)$ with $K := K(\mathcal{E})$. After translation, $f = f' = (0,0)$ holds. Since g, h as well as g', h' are linearly independent, there exists an $A \in \mathrm{GL}(2, K)$ with $Ag = g'$ and $Ah = h'$. The claim now follows from Theorem 3.9. ☐

Theorem 4.10. *Desarguesian planes $\mathcal{E}$ and $\mathcal{E}'$ are isomorphic if and only if the skew fields $K := K(\mathcal{E})$ and $K' := K(\mathcal{E}')$ are isomorphic.*

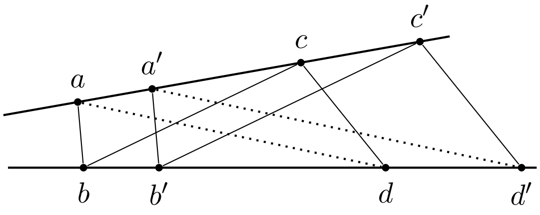
Proof. By the Theorem of Desargues, we can assume $\mathcal{E} = \mathcal{E}(K)$ and $\mathcal{E}' = \mathcal{E}(K')$. If $\varphi\colon K \to K'$ is an isomorphism, then $K^2 \to (K')^2$, $(x, y) \mapsto (\varphi(x), \varphi(y))$ is an isomorphism between $\mathcal{E}$ and $\mathcal{E}'$, because

$$\varphi(a + Kv) = \varphi(a) + \varphi(K)\varphi(v) = \varphi(a) + K\varphi(v)$$

for $a, v \in K^2$. Conversely, let $\varphi\colon K^2 \to (K')^2$ be an isomorphism between $\mathcal{E}$ and $\mathcal{E}'$. By Theorem 3.9, we can assume $\varphi(0) = 0$ and $\varphi(e_i) = e_i$ for $i = 1, 2$. As in the proof of Theorem 3.9, there exists a ring isomorphism $\alpha\colon K \to K'$ with $\varphi(x, y) = (\alpha(x), \alpha(y))$ for all $x, y \in K$. In particular, K and K' are isomorphic. ☐

Theorem 4.11 (Shear Theorem)**.** *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Desarguesian plane. For pairwise distinct points $a, a', c, c' \in G \in \mathcal{G}$ and $b, b', d, d' \in H \in \mathcal{G}$, it holds that*

$$ab \parallel a'b',\ bc \parallel b'c',\ cd \parallel c'd' \implies ad \parallel a'd'.$$



Proof. By the Little Shear Theorem, we may assume $G \wedge H = \mathbf{o}$. In the case $a = \mathbf{o}$, it would follow that $H = ab \parallel a'b'$ and $G = aa' = H$. Therefore $a \neq \mathbf{o}$ and there exists an $\alpha \in K(\mathcal{E})$ with $a' = \alpha(a)$. From the Intercept Theorem, it follows that $b' = \alpha(b)$, $c' = \alpha(c)$, $d' = \alpha(d)$ and $ad \parallel a'd'$. ☐

Remark 4.12. In order to construct translation planes that are not Desarguesian planes, we weaken the concept of a skew field.

Definition 4.13. A set Q with binary operations $+$ and $\cdot$ is called a *quasifield*, if the following hold:

(Q1) $(Q, +)$ is an abelian group with neutral element 0.

(Q2) For all $x \in Q$, $x \cdot 0 = 0 = 0 \cdot x$ holds.

(Q3) There exists $1 \in Q \setminus \{0\}$ with $1 \cdot x = x = x \cdot 1$ for all $x \in Q$.

(Q4) For $a, b \in Q \setminus \{0\}$, $a \cdot b \neq 0$ and there exist uniquely determined $x, y \in Q \setminus \{0\}$ with $a \cdot x = b = y \cdot a$.

(Q5) For all $x, y, z \in Q$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ holds.

(Q6) For all $a, b, c \in Q$ with $a \neq b$, there exists exactly one $x \in Q$ with $a \cdot x = (b \cdot x) + c$.

Remark 4.14. As usual, we use the shorthand notation $xy - z := (x \cdot y) + (-z)$ for elements of a quasifield. Since in (Q5) only the left distributive law is required ((Q6) is a replacement for the right distributive law), one should more precisely speak of a left quasifield. Obviously, every skew field is a quasifield.

Theorem 4.15 (Hall). *Let Q be a quasifield. For $x, a, b \in Q$, let $G_x := \{x\} \times Q$ and $G_{a,b} := \{(x, y) \in Q^2 : ax + b = y\}$. Then*

$$\mathcal{E} = \big(Q^2, \{G_x, G_{a,b} : x, a, b \in Q\}\big)$$

is a translation plane.

Proof. Because of $0, 1 \in Q$, every line has at least two points. Let $(x, y), (x', y') \in Q^2$ be distinct. In the case $x = x'$, the points lie only on the "vertical" line G_x, because otherwise $y = ax + b = ax' + b = y'$. In the case $x \neq x'$, there exists exactly one $a \in Q$ with $ax - ax' = a(x - x') = y - y'$ (for $y = y'$, $a = 0$). For $b := y - ax$, it then holds that $ax + b = y$ and $ax' + b = ax + y' - y + b = y'$. Conversely, a and b are uniquely determined by these equations. Thus, the points lie only on $G_{a,b}$. Let $(x, y) \in Q^2$ and a line G be given. In the case $G = G_{x'}$, (x, y) lies only on the parallel G_x of G. So let $G = G_{a,b}$ and $b' := y - ax \in Q$. Then $ax + b' = y$ holds, i. e. (x, y) lies on $G_{a,b'} \parallel G$. For $a' \neq a$ and $c \in Q$, $G_{a,b} \cap G_{a',c} \neq \varnothing$ holds according to (Q6). Therefore, $G_{a,b'}$ is the only parallel of G that contains (x, y). This shows the parallel axiom. Finally, the points $(0, 1)$, $(1, 0)$, and $(1, 1)$ are in general position. Thus, $\mathcal{E}$ is an affine plane.

Let $(s, t) \in Q^2$ be arbitrary. To prove the translation plane, it suffices to construct a translation φ with $\varphi(0, 0) = (s, t)$. For $(x, y) \in Q^2$, let $\varphi(x, y) := (x + s, y + t)$. Since $(Q, +)$ is an abelian group, $\varphi\colon Q^2 \to Q^2$ is a bijection with $\varphi(0, 0) = (s, t)$. Furthermore, φ has fixed points only if $\varphi = \mathrm{id}$. Obviously, $\varphi(G_x) = G_{x+s} \parallel G_x$ holds. For $(x, y) \in G_{a,b}$ and $b' := b - as + t$, it holds that $a(x + s) + b' = ax + as + b - as + t = y + t$. Since b' does not depend on (x, y), it follows that $\varphi(G_{a,b}) = G_{a,b'}$. Thus, φ is a translation and $\mathcal{E}$ is a translation plane. $\square$

Remark 4.16. For a Galois extension $K \subseteq L$ with $G := \mathrm{Gal}(L|K)$, let $N\colon L \to L$, $x \mapsto \prod_{g \in G} g(x)$ be the norm map. For $x, y \in L$ and $g \in G$, it obviously holds that $N(xy) = N(x)N(y)$ and $N(g(x)) = N(x)$.

Theorem 4.17 (André). *Let $K \subseteq L$ be a Galois extension and $G := \mathrm{Gal}(L|K)$ and $\varphi\colon L \to G$ be an arbitrary map with $\varphi(0) = \varphi(1) = 1$. With the multiplication*

$$x * y := x\varphi(N(x))(y) \qquad (x, y \in L)$$

$(L, +, *)$ becomes a quasifield.

Proof. The axioms (Q1)–(Q3) hold by definition. For $x, y, z \in Q$ we have

$$x * (y + z) = x\varphi(N(x))(y + z) = x\varphi(N(x))(y) + x\varphi(N(x))(z) = x * y + x * z,$$

thus (Q5).

Let $a, b, c \in Q$ with $a \neq b$. In the case $\sigma := \varphi(N(a)) = \varphi(N(b))$, $a * x = b * x + c$ is equivalent to $(a - b)\sigma(x) = c$. Then (Q6) holds. Therefore, let $\tau := \varphi(N(b)) \neq \sigma$ and thus $N(a) \neq N(b)$. Wlog. let $a \neq 0$. After multiplication by a^{-1} in L we obtain $\sigma(x) - b\tau(x) = c$ with $N(b) \neq 1$. Finally, we can assume $\tau = 1$. It suffices to show that the K-linear map $f\colon L \to L$, $x \mapsto \sigma(x) - bx$ is invertible. Suppose indirectly that $x \in L$ is an eigenvector of σ for the eigenvalue b. Let $n := |\langle\sigma\rangle|$. Then

$$x = \sigma^n(x) = \sigma^{n-1}(bx) = \sigma^{n-2}(\sigma(b)bx) = \ldots = x\prod_{i=1}^{n}\sigma^i(b).$$

If $\gamma_1, \ldots, \gamma_r \in G$ is a system of representatives for the cosets of $\langle\sigma\rangle$ in G, then we obtain the contradiction

$$N(b) = \prod_{i=1}^{r}\prod_{j=1}^{n}\gamma_i\sigma^j(b) = \prod_{i=1}^{r}\gamma_i(1) = 1.$$

Thus (Q6) holds.

It remains to show (Q4). For $a, b \in Q \setminus \{0\}$, certainly $a * b \neq 0$. Furthermore, there exists exactly one $x \in Q$ with $a * x = a\varphi(N(a))(x) = b$. We set $y := b\varphi(N(a^{-1}b))(a)^{-1}$, where $^{-1}$ denotes the inverse in $(L, \cdot)$. Because of $N(y) = N(a^{-1}b)$ (Remark 4.16), it holds that

$$y * a = b\varphi(N(y))(a)^{-1}\varphi(N(y))(a) = b.$$

Now let $y * a = b = z * a$. With Remark 4.16 it follows that

$$N(y)N(a) = N(y\varphi(N(y))(a)) = N(y * a) = N(z)N(a)$$

and $N(y) = N(z)$. From this one obtains $y = (y * a)\varphi(N(y))(a)^{-1} = (z * a)\varphi(N(z))(a)^{-1} = z$. Thus (Q4) holds. □

Theorem 4.18. *There exists a translation plane of order* 9 *that is not a Desarguesian plane.*

Proof. We consider the Galois extension $\mathbb{F}_3 \subseteq \mathbb{F}_9$ with $G := \mathrm{Gal}(\mathbb{F}_9|\mathbb{F}_3) = \langle\sigma\rangle \cong C_2$ (it holds that $\sigma(x) = x^3$ for $x \in \mathbb{F}_9$). For $a \in \mathbb{F}_9$ we have $N(a) = a\sigma(a) = a^4$ and $N(\mathbb{F}_9) = \mathbb{F}_3$. We define $\varphi(-1) := \sigma$ in Theorem 4.17 and obtain the quasifield Q. For $\mathbb{F}_9^\times = \langle\zeta\rangle$ it holds that

$$\zeta^i * \zeta^j = \begin{cases} \zeta^{i+j} & \text{if } i \equiv 0 \pmod 2, \\ \zeta^{i+3j} & \text{if } i \equiv 1 \pmod 2. \end{cases}$$

From Theorem 4.15 one obtains a translation plane $\mathcal{E}$ with points Q^2. In Definition 3.4 we choose $\mathbf{o} = (0, 0)$, so that the addition in $\mathcal{E}$ coincides with the component-wise addition in Q^2. Let $\gamma \in K(\mathcal{E}) \subseteq \mathrm{End}(Q^2)$. Because of $\gamma(1, 0) \in G_{0,0}$ there exists $s \in Q$ with $\gamma(1, 0) = (s, 0)$. Because of $\gamma(0, x) \in G_0$ and $(s, 0) + \gamma(0, x) = \gamma(1, x) \in G_{x,0}$ we have $\gamma(1, x) = (s, x * s)$ and $\gamma(0, x) = (0, x * s)$ for all $x \in Q$. From $\gamma(x, 0) + (0, x * s) = \gamma(x, x) \in G_{1,0}$ it follows that $\gamma(x, 0) = (x * s, 0)$ for all $x \in Q$. Thus γ is uniquely determined by s. Because of $\gamma \in \mathrm{End}(Q^2)$ it holds that $(x + y) * s = x * s + y * s$ for $x, y \in Q$. From $\zeta + \zeta^3 \in \mathbb{F}_3$ it follows that $(\zeta + \zeta^3)s = \zeta\sigma(s) + \zeta^3\sigma(s)$ and $\sigma(s) = s \in \mathbb{F}_3$. This shows $|K(\mathcal{E})| \leq 3$ and $\mathcal{E}$ cannot be a Desarguesian plane according to Theorem 4.8. □

Remark 4.19. The plane constructed in Theorem 4.18 is called a *Hall plane*. Besides $\mathcal{E}(\mathbb{F}_9)$ there are, up to isomorphism, two further affine planes of order 9, which however are not translation planes (without proof).

Example 4.20. Let $\mathbb{H} := \mathbb{C} + \mathbb{C}j$ be the Hamiltonian skew field with $j^2 = -1$ and $ij = -ji$. For $z := x + yj \in \mathbb{H}$ let $z^* := \overline{x} - yj$. It holds that

$$zz^* = x\overline{x} + y\overline{y} = z^*z \in \mathbb{R}.$$

For $z_1, z_2 \in \mathbb{H}$ we have $(z_1 + z_2)^* = z_1^* + z_2^*$ and

$$\begin{aligned}(z_1z_2)^* &= (x_1x_2 - y_1\overline{y_2} + (x_1y_2 + y_1\overline{x_2})j)^* = \overline{x_1x_2} - \overline{y_1}y_2 - (x_1y_2 + y_1\overline{x_2})j\\ &= (\overline{x_2} - y_2j)(\overline{x_1} - y_1j) = z_2^*z_1^*.\end{aligned}$$

We consider $\mathbb{O} := \mathbb{H}^2$ with the multiplication

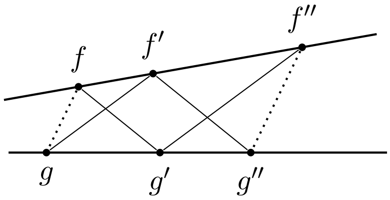
$$(a, b) \cdot (c, d) := (ac - d^*b, da + bc^*) \qquad (a, b, c, d \in \mathbb{H}).$$

One can show that $\mathbb{O}$ is a quasifield. Its elements are called *octonions*.

5 Pappus Planes

Definition 5.1. An affine plane is called a *Pappus plane*, if for $F, G \in \mathcal{G}$ and pairwise distinct points $f, f', f'' \in F$ and $g, g', g'' \in G$ it holds that

$$fg' \parallel f'g'',\ f'g \parallel f''g' \implies fg \parallel f''g''.$$



Theorem 5.2 (Hessenberg). *Every Pappus plane is a Desargues plane and therefore a translation plane.*

Proof. Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Pappus plane and $F, G, H \in \mathcal{G}$ with $x = F \wedge G = F \wedge H = G \wedge H$. Let $f, f' \in F \setminus \{x\}$, $g, g' \in G \setminus \{x\}$, $h, h' \in H \setminus \{x\}$ with $fg \parallel f'g'$ and $gh \parallel g'h'$. In the case $f = f'$, it follows that $g = g'$ and $h = h'$. Then $fh = f'h'$ holds. We can therefore assume that no two points are equal. In the case $fh \parallel G \parallel f'h'$, we are finished. Wlog. let therefore $G \nparallel f'h'$. Let G' be the parallel to G through h. Then there exists

$$k := G' \wedge f'h'.$$

Because of $h \notin G$ and $G \nparallel H$, it follows that $k \notin \{g', h'\}$. Let temporarily $gh \parallel kg'$. Then $k \in g'h'$ because of $gh \parallel g'h'$. By definition, $k \in f'h'$ also holds, and it follows that $g'h' = kh' = f'h'$. Thus $f'h' = f'g'$ and $fh = fg \parallel f'g' = f'h'$ as desired. Let therefore $gh \nparallel kg'$ and

$$l := gh \wedge kg'.$$

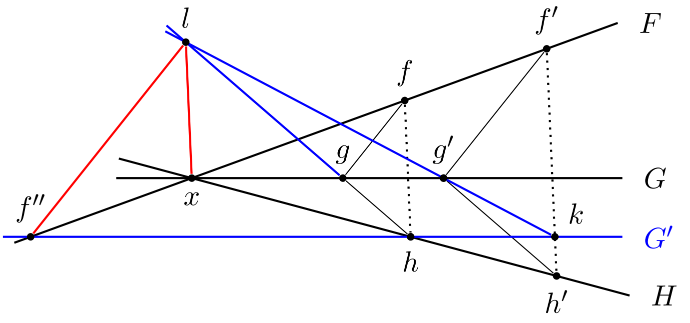
We apply the Pappus property three times:

(i) The points x, h, h' and l, g', k are each collinear. In the case $l = h$, it would be $h = hg' \wedge G' = k \in f'h'$ and $f' \in H$. Thus $hl \parallel h'g'$ and $xg' \parallel hk$. It follows that $xl \parallel h'k$.

(ii) Let $f'' := G' \wedge F$. The points x, f, f'' and l, g', k are each collinear. In the case $f'' = l$, it would be $h \in gh \cap G' = l = f'' \in F$. In the case $f' = k$, it is $k = G' \wedge F = f''$ and $f''l = f'l = f'g' \parallel fg$. Otherwise, $xl \parallel h'k = f'k$ and $xg' \parallel f''k$ hold. It follows that $f''l \parallel f'g' \parallel fg$.

(iii) The points l, g, h are collinear. From $f''l \parallel fg$ and $f''h \parallel xg$, it follows that $xl \parallel fh$.

Altogether, $fh \parallel xl \parallel h'k \parallel f'h'$.



The second statement follows from Theorem 4.3. □

Remark 5.3. Analogously to Desargues, the Pappus property implies the validity of the little theorem of Pappus.

Theorem 5.4 (Pappus)**.** *For every affine plane $\mathcal{E}$, the following are equivalent:*

(1) $\mathcal{E}$ is a Pappus plane.

(2) $\mathcal{E}$ is isomorphic to a coordinate plane of a field.

Proof. (1)⇒(2): According to Hessenberg, $\mathcal{E}$ is a Desarguesian plane and as such isomorphic to the coordinate plane $\mathcal{E}(K)$ of a skew field K (Theorem 4.8). Let $\alpha, \beta \in K \backslash \{0, 1\}$. We consider $f = (1, 0)$ and $g = (0, 1)$. It holds that $f\alpha(g) \parallel \alpha^{-1}(f)g$ as well as $\beta^{-1}(f)g \parallel f\beta(g)$. The Pappus property shows $\alpha^{-1}(f)\beta(g) \parallel \beta^{-1}(f)\alpha(g)$. Therefore, there exists $\gamma \in K$ with

$$\alpha^{-1}(f) - \beta(g) = \gamma(\beta^{-1}(f) - \alpha(g)).$$

Since f and g are linearly independent, one obtains $\alpha^{-1} = \gamma\beta^{-1}$ and $\beta = \gamma\alpha$. This shows $\alpha\beta = \alpha\gamma\alpha = \beta\alpha$. Thus K is a field.

(2)⇒(1): According to the little theorem of Pappus, we may assume that the lines F and G in the Pappus property are not parallel. After translation, they intersect in $\mathbf{o} = (0, 0) \in K^2$. There exist $\alpha, \beta \in K \setminus \{0, 1\}$ with $f' = \alpha(f)$ and $f'' = \beta(f)$. From $fg' \parallel \alpha(f)g''$ and $g\alpha(f) \parallel g'\beta(f)$ it follows that $g'' = \alpha(g')$ as well as $\alpha^{-1}\beta(g) = \beta\alpha^{-1}(g) = g'$. Thus one obtains

$$fg = f\beta^{-1}\alpha(g') \parallel \beta(f)\alpha(g') = f''g''.$$ □

Remark 5.5.

(i) Since the Hamiltonian skew field $\mathbb{H}$ is not a field, $\mathcal{E}(\mathbb{H})$ is a Desarguesian plane, but not a Pappus plane. Note:

$$\text{Pappus plane} \Longrightarrow \text{Desarguesian plane} \Longrightarrow \text{translation plane} \Longrightarrow \text{affine plane.}$$

(ii) Wedderburn proved that every finite skew field is commutative.[1] Therefore, every finite Desar-

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[1] See Algebra notes

guesian plane is a Pappus plane.

Lemma 5.6. *Let K be a field, $f, g \in K^2$ linearly independent and $\alpha, \beta, \gamma, \delta \in K$ with $\alpha\delta \neq \beta\gamma$. Then*

$$(\alpha f)(\beta g) \wedge (\gamma f)(\delta g) = \frac{1}{\alpha\delta - \beta\gamma}(\alpha\delta(\gamma f + \beta g) - \beta\gamma(\alpha f + \delta g)).$$

Proof. From the assumption, it follows that $\alpha f \neq \beta g$ and $\gamma g \neq \delta g$. A common point on $(\alpha f)(\beta g)$ and $(\gamma f)(\delta g)$ can be written in the form

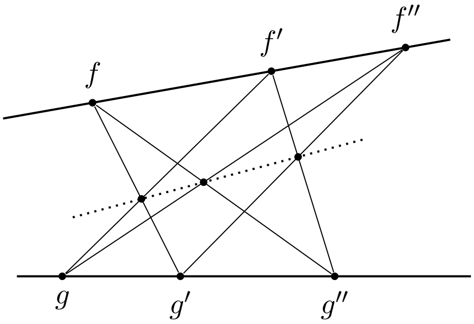
$$x = \lambda\alpha f + (1 - \lambda)\beta g = \mu\gamma f + (1 - \mu)\delta g$$

with $\lambda, \mu \in K$. Since f and g are linearly independent, it holds that $\lambda\alpha = \mu\gamma$ and $(1-\lambda)\beta = (1-\mu)\delta$. It follows

$$\lambda = \frac{\gamma(\delta - \beta)}{\alpha\delta - \beta\gamma}, \qquad \mu = \frac{\alpha(\delta - \beta)}{\alpha\delta - \beta\gamma}.$$

In particular, x is uniquely determined. ☐

Theorem 5.7 (Pascal)**.** *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a Pappian plane and $F, G \in \mathcal{G}$. Let $f, f', f'' \in F \setminus G$ and $g, g', g'' \in G \setminus F$ be pairwise distinct, such that the intersection points $x := fg' \wedge f'g$, $y := f'g'' \wedge f''g'$ and $z := fg'' \wedge f''g$ exist. Then x, y, z are collinear.*



Proof. As usual, let $\mathcal{E} = \mathcal{E}(K)$ for a field K. Let us first consider the case that F and G intersect in a point, wlog. 0. By assumption, there exist $\alpha, \beta, \gamma, \delta \in K \setminus \{0, 1\}$ with $f' = \alpha f$, $f'' = \beta f$, $g' = \gamma g$ and $g'' = \delta g$. With $u := f + g$, $v := \alpha f + \gamma g$ and $w := \beta f + \delta g$ one obtains

$$x = \frac{1}{\alpha\gamma - 1}(\alpha\gamma u - v), \qquad y = \frac{1}{\beta\delta - \alpha\gamma}(\beta\delta v - \alpha\gamma w), \qquad z = \frac{1}{1 - \beta\delta}(w - \beta\delta u)$$

from Lemma 5.6. Because of

$$\begin{aligned} \beta\delta(\alpha\gamma - 1)x + (\beta\delta - \alpha\gamma)y + \alpha\gamma(1 - \beta\delta)z = 0, \\ \beta\delta(\alpha\gamma - 1) + (\beta\delta - \alpha\gamma) + \alpha\gamma(1 - \beta\delta) = 0 \end{aligned}$$

it holds that $y = \lambda x + (1 - \lambda)z$ for some $\lambda \in K$. Thus x, y, z are collinear.

Now let $F \parallel G$. After translation, let $z = (0, 0)$. We describe all points with respect to the basis f, f''. According to the intercept theorem, there exist $\alpha, \gamma, \delta \in K$ with $g'' = \alpha f$, $g = \alpha f''$, $x =$

$\gamma f+(1-\gamma)g' = \gamma f'+(1-\gamma)g$, $y = \delta f'+(1-\delta)g'' = \delta f''+(1-\delta)g'$. Furthermore, there exist $\lambda,\mu \in K$ with $f' = \lambda f+(1-\lambda)f''$ as well as $g' = \mu g+(1-\mu)g'' = \mu\alpha f''+(1-\mu)\alpha f$. It follows

$$\begin{aligned} x &= (\gamma+(1-\gamma)(1-\mu)\alpha)f+(1-\gamma)\mu\alpha f'' = \gamma\lambda f+(\gamma(1-\lambda)+(1-\gamma)\alpha)f'' \\ y &= (\delta\lambda+(1-\delta)\alpha)f+\delta(1-\lambda)f'' = (1-\delta)(1-\mu)\alpha f+(\delta+(1-\delta)\mu\alpha)f'' \end{aligned}$$

A comparison of coefficients yields

$$\begin{aligned} \gamma+(1-\gamma)(1-\mu)\alpha = \gamma\lambda \qquad &\Longrightarrow \qquad \frac{\gamma}{1-\gamma} = -\frac{(1-\mu)\alpha}{1-\lambda} \\ \delta(1-\lambda) = \delta+(1-\delta)\mu\alpha \qquad &\Longrightarrow \qquad \frac{\delta}{1-\delta} = -\frac{\mu\alpha}{\lambda} \end{aligned}$$

One now sees that the coefficients of x and y divide in the same ratio:

$$\frac{\gamma\lambda}{(1-\gamma)\mu\alpha} = -\frac{(1-\mu)\lambda}{\mu(1-\lambda)} = \frac{(1-\delta)(1-\mu)\alpha}{\delta(1-\lambda)}$$

Thus x, y are linearly dependent and $x, y, z = (0, 0)$ are collinear. ☐

Remark 5.8. In Theorem 5.7, $xy = xz$ is called the *Pascal line*. In the following, let K always be a field.

Definition 5.9. For row vectors $x, y, z \in K^2$ let

$$[x,y] := \det\begin{pmatrix} x \\ y \end{pmatrix}, \qquad [x,y,z] := \det\begin{pmatrix} 1 & x \\ 1 & y \\ 1 & z \end{pmatrix}.$$

Lemma 5.10. *For $x, x', y, y', z, z' \in K^2$ it holds that*

$$\begin{aligned} &[x,y,z] = [y,z,x] = [z,x,y] = [x-z,y-z] = [x,y]+[y,z]+[z,x], \\ &[x+x',y+y',z+z'] = [x,y,z]+[x,y',z']+[x',y,z']+[x',y',z], \\ &[x,y]z+[y,z]x+[z,x]y = 0. \end{aligned}$$

Proof. The first line follows from the determinant rules and Gaussian elimination. For the second line, one uses the first:

$$\begin{aligned} [x+x',y+y',z+z'] &= [x,y]+[x,y']+[x',y]+[x',y']+[y,z]+[y,z'] \\ &\quad +[y',z]+[y',z']+[z,x]+[z,x']+[z',x]+[z',x'] \\ &= [x,y,z]+[x,y',z']+[x',y,z']+[x',y',z]. \end{aligned}$$

If x, y are linearly dependent, say $x = \lambda y$ with $\lambda \in K$, then $[y,z]x+[z,x]y = (\lambda[y,z]+\lambda[z,y])y = 0$ holds. If x, y are linearly independent, then $z = \lambda x+\mu y$ with $\lambda,\mu \in K$ holds and it follows that

$$[x,y]z+[y,z]x+[z,x]y = (\lambda[x,y]+\lambda[y,x])x+(\mu[x,y]+\mu[y,x])y = 0.$$ ☐

Lemma 5.11. *$x, y, z \in K^2$ are collinear if and only if $[x,y,z] = 0$.*

Proof. It holds:

$$x, y, z \text{ collinear} \iff x-z, y-z \text{ linearly dependent} \overset{5.10}{\iff} [x,y,z] = [x-z, y-z] = 0. \qquad \square$$

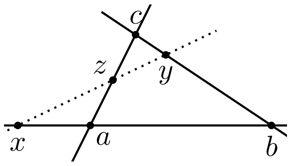
Theorem 5.12. *Let $a, b, c \in K^2$ be in general position and*

$$x := \alpha a + (1-\alpha)b, \qquad y := \beta b + (1-\beta)c, \qquad z := \gamma c + (1-\gamma)a$$

be points on the lines ab, bc and ca, where $\alpha, \beta, \gamma \in K \setminus \{1\}$.

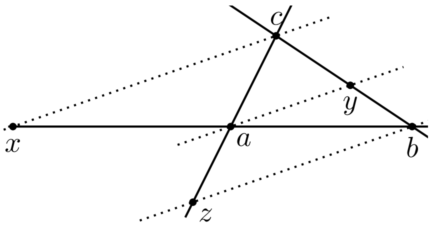
(i) (Menelaus) *The points x, y, z are collinear if and only if*

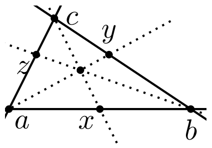
$$\frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\frac{\gamma}{1-\gamma} = -1.$$



(ii) (Ceva) *The lines ay, bz and cx are parallel or intersect in a point if and only if*

$$\frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\frac{\gamma}{1-\gamma} = 1.$$





Proof. After translation, we can assume $a = (0,0)$. Then b, c is a basis of K^2. We represent x, y, z with respect to this basis: $x = (1-\alpha, 0)$, $y = (\beta, 1-\beta)$, $z = (0, \gamma)$.

(i) From Sarrus' rule for determinants, it follows that

$$[x,y,z] = \beta\gamma + (1-\alpha)(1-\beta) - \gamma(1-\alpha).$$

Therefore,

$$[x,y,z] = 0 \iff \alpha\beta + \beta\gamma + \gamma\alpha + 1 = \alpha + \beta + \gamma \iff \frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\frac{\gamma}{1-\gamma} = -1.$$

(ii) If ay, bz, cx are parallel, then $y = (\beta, 1-\beta)$, $b - z = (1, -\gamma)$, and $c - x = (\alpha - 1, 1)$ are pairwise linearly dependent. This leads to $1 - \beta = -\beta\gamma$ and $1 = \gamma(1-\alpha)$. It follows that

$$\frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\frac{\gamma}{1-\gamma} = (\gamma - 1)\frac{-1}{\gamma}\frac{\gamma}{1-\gamma} = 1.$$

If the lines intersect in a point, then there exist $\lambda, \mu, \rho \in K$ with

$$(\lambda\beta, \lambda(1-\beta)) = \lambda y = \mu x + (1-\mu)c = (\mu(1-\alpha), 1-\mu) = \rho z + (1-\rho)b = (1-\rho, \rho\gamma).$$

It follows that $\lambda\beta = \mu(1-\alpha) = 1-\rho$ and $\lambda(1-\beta) = 1-\mu = \rho\gamma$. This yields

$$\frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\frac{\gamma}{1-\gamma} = \frac{\mu\alpha}{\lambda\beta}\frac{\lambda\beta}{\rho\gamma}\frac{\rho\gamma}{\rho(1-\gamma)} = \frac{\mu\alpha}{\rho-1+\mu} = 1.$$

Conversely, let

$$\frac{\alpha}{1-\alpha}\frac{\beta}{1-\beta}\frac{\gamma}{1-\gamma} = 1.$$

In the case $ay \parallel bz$, we have $1-\beta = -\beta\gamma$ and it follows that $\alpha = (1-\alpha)(\gamma-1)$. This yields $\gamma(1-\alpha) = 1$ and $ay \parallel bz \parallel cx$. Now let $ay \nparallel bz$. Then there exist $\lambda, \rho \in K$ with

$$(\lambda\beta, \lambda(1-\beta)) = \lambda y = \rho z + (1-\rho)b = (1-\rho, \rho\gamma).$$

Thus $\lambda\beta = 1-\rho$ and $\lambda(1-\beta) = \rho\gamma$ hold. It follows that $\alpha(1-\rho) = (1-\alpha)\rho(1-\gamma)$ and

$$\alpha(1-\rho\gamma) = \alpha(1-\lambda+\lambda\beta) = \alpha(2-\rho-\gamma) = \rho(1-\gamma).$$

With $\mu := 1-\rho\gamma$, it now holds that $\mu(1-\alpha) = 1-\rho\gamma-\rho+\rho\gamma = 1-\rho$. As above, ay, bz, and cx now have a common point of intersection. $\square$

Remark 5.13.

(i) In the situation of Theorem 5.12, $xy = xz$ is called the *Menelaus line* and $ay \wedge bz = ay \wedge cx$ is called the *Ceva point*.

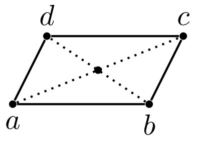
(ii) If K has characteristic $\operatorname{char} K = 2$, then the conditions of Menelaus and Ceva are identical, i.e., x, y, z are collinear if and only if $ay \parallel bz \parallel cx$ or $ay \wedge bz = ay \wedge cx$. This provides a geometric characterization of the field property $\operatorname{char} K = 2$.

Definition 5.14. Let $\operatorname{char} K \neq 2$. For distinct $a, b \in K^2$, the *midpoint* between a and b can then be defined by $\frac{1}{2}(a+b)$.

Theorem 5.15 (Diagonal Theorem)**.** *Let* $\operatorname{char} K \neq 2$. *Let* $a, b, c, d \in K^2$ *be pairwise distinct and non-collinear. Then the following statements are equivalent:*

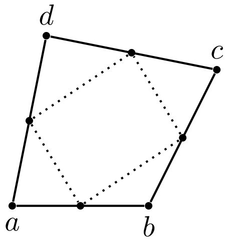
(1) a, b, c, d *form a parallelogram, i. e.* $ab \parallel cd$ *and* $ad \parallel bc$.

(2) The diagonals ac *and* bd *intersect at the midpoint* $\frac{1}{2}(a+c) = \frac{1}{2}(b+d)$.



Proof. Follows from the Parallelogram Theorem. $\square$

Theorem 5.16 (Varignon)**.** *Let* $\operatorname{char} K \neq 2$*. Let* $a, b, c, d \in K^2$ *be pairwise distinct and non-collinear. Then the midpoints of the sides* $\frac{1}{2}(a+b)$*,* $\frac{1}{2}(b+c)$*,* $\frac{1}{2}(c+d)$*,* $\frac{1}{2}(d+a)$ *form a parallelogram.*



Proof. In the case $a + b = c + d$, we have $\frac{1}{4}(a+d) + \frac{1}{4}(b+c) = \frac{1}{2}(a+b)$ and all four midpoints lie on a line, which we interpret as a parallelogram. Now let the midpoints be pairwise distinct and non-collinear. Then the assertion follows from the Diagonal Theorem. ☐

Theorem 5.17. *Let* $\operatorname{char} K \neq 2$ *and* $a, b, c \in K^2$ *in general position. Then the following statements are equivalent:*

(1) $\operatorname{char} K \neq 3$.

(2) The lines $\frac{1}{2}(a+b)c$*,* $\frac{1}{2}(b+c)a$*,* $\frac{1}{2}(a+c)b$ *intersect in a point.*

Proof. If two of the lines are equal, say $\frac{1}{2}(a+b)c = \frac{1}{2}(b+c)a$, then there exists $\lambda \in K$ with $c = a + \lambda(b + c - 2a)$. Then a, b, c would be collinear according to Lemma 5.11. Thus, the lines are always pairwise distinct.

(1)⇒(2): One easily sees that $\frac{1}{3}(a+b+c)$ lies on all three lines.

(2)⇒(1): Let s be the common intersection point. After translation, we can assume $c = (0,0)$. Then there exist $\lambda, \mu \in K$ with

$$s = \frac{\lambda}{2}(a+b) = \frac{\mu}{2}b - (1-\mu)a.$$

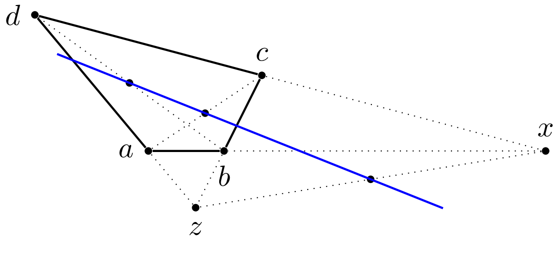
A comparison of coefficients yields $\lambda = \mu$ and $\lambda/2 = (1-\lambda)$. This shows $3\lambda = 2 \neq 0$, hence $\operatorname{char} K \neq 3$. ☐

Definition 5.18. Four points a, b, c, d of an affine plane form a *general quadrilateral*, if no three of the points are collinear and no two of the possible connecting lines are parallel. The intersection points

$$x := ab \wedge cd, \qquad y := ac \wedge bd, \qquad z := ad \wedge bc$$

are called *diagonal points* of the quadrilateral.

Theorem 5.19 (Gauss)**.** *Let* $\operatorname{char} K \neq 2$*. Let* a, b, c, d *be a general quadrilateral in* $\mathcal{E}(K)$ *with diagonal points* x, y, z*. Then the midpoints of ac, bd and xz lie on a line, which is called the* Gauss line*.*



Proof. From Lemma 5.10 and Lemma 5.11 it follows that

$$4[\frac{1}{2}(a+c), \frac{1}{2}(b+d), \frac{1}{2}(x+z)] = [a,b,x] + [a,d,z] + [c,b,z] + [c,d,x] = 0. \qquad \square$$

6 Projective Planes

Definition 6.1. A pair $(\mathcal{P}, \mathcal{G})$ is called a *projective plane*, if the following hold:

(P1) Every line contains at least three distinct points.

(P2) Two distinct points lie on exactly one line.

(P3) Any two lines have an intersection point.

(P4) There exist three points in general position.

Remark 6.2. In contrast to affine planes, parallel lines of a projective plane are already identical. Due to (P2), (P3) is therefore equivalent to: Any two distinct lines have exactly one intersection point.

Theorem 6.3. *For every affine plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$,*

$$\overline{\mathcal{E}} := \big(\mathcal{P} \cup [\mathcal{G}], \{G \cup \{[G]\} : G \in \mathcal{G}\} \cup \{[\mathcal{G}]\}\big)$$

is a projective plane. One calls $\overline{\mathcal{E}}$ the projective closure *of $\mathcal{E}$.*

Proof. Since $G \in \mathcal{G}$ contains at least two points, $G \cup \{[G]\}$ possesses at least three points (note that $[G]$ is a point of $\overline{\mathcal{E}}$). Since $\mathcal{E}$ possesses three points in general position, $|[\mathcal{G}]| \geq 3$ holds. This shows (P1).

Distinct $x, y \in \mathcal{P}$ lie in exactly one line $G \cup \{[G]\}$ of $\overline{\mathcal{E}}$ (note that $\{[G]\} \cap \mathcal{P} = \varnothing = [\mathcal{G}] \cap \mathcal{P}$). Distinct $[G], [H] \in [\mathcal{G}]$ lie only in the line $[\mathcal{G}]$. Finally, let $x \in \mathcal{P}$ and $[G] \in [\mathcal{G}]$. According to the parallel axiom, there exists exactly one $H \in [G]$ with $x \in H$. Thus x, $[G] = [H]$ lie only in the line $H \cap \{[H]\}$. Therefore (P2) holds.

Let $G \cup \{[G]\}$ and $H \cup \{[H]\}$ be distinct lines in $\overline{\mathcal{E}}$. If G and H are parallel, then $[G] = [H]$ is an intersection point. Otherwise $G \wedge H$ is an intersection point. This shows (P3).

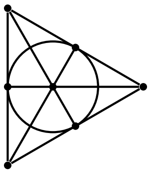
Three points from $\mathcal{E}$ in general position are obviously also in general position in $\overline{\mathcal{E}}$. Thus (P4) holds. $\square$

Remark 6.4. Let $\varphi \colon \mathcal{E} \to \mathcal{E}'$ be an isomorphism of affine planes. Because of $G \parallel H \iff \varphi(G) \parallel \varphi(H)$ (Remark 2.2), φ extends to an isomorphism $\overline{\mathcal{E}} \to \overline{\mathcal{E}'}$.

Example 6.5. Let $\mathcal{E} := \mathcal{E}(K)$ be the coordinate plane of a skew field K. The projective closure of $\mathcal{E}$ then contains, in addition to the points from K^2, the directions Kv for $v \in K^2 \setminus \{0\}$ (i.e., the 1-dimensional subspaces of K^2). Both sets can be described uniformly by the 1-dimensional subspaces of K^3. The elements from K^2 correspond to the subspaces of the form $K(x, y, 1)$ and the points $K(x, y)$ correspond to $K(x, y, 0)$. The lines of $\overline{\mathcal{E}}$ are now exactly the 2-dimensional subspaces of K^3. The intersection point of two distinct lines G, H in $\overline{\mathcal{E}}$ is the 1-dimensional space $G \cap H$. Furthermore

$$\mathrm{PGL}(3, K) := \mathrm{GL}(3, K)/K^{\times}1_3 \leq \mathrm{Aut}(\overline{\mathcal{E}}).$$

Therefore, $\mathrm{Aut}(\overline{\mathcal{E}})$ is significantly larger than $\mathrm{Aut}(\mathcal{E})$, i.e., $\overline{\mathcal{E}}$ is "more symmetric" than $\mathcal{E}$. One calls $\overline{\mathcal{E}}$ the *projective coordinate plane* over K. In the smallest case $K = \mathbb{F}_2$, one calls $\overline{\mathcal{E}}$ the *Fano plane*.



Theorem 6.6. *For every projective plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ and $F \in \mathcal{G}$,*

$$\mathcal{E}_F := \big(\mathcal{P} \setminus F, \{G \setminus F : G \in \mathcal{G} \setminus \{F\}\}\big)$$

is an affine plane.

Proof. According to (P1) and Remark 6.2, $|G \setminus F| \geq 2$ holds for $G \in \mathcal{G} \setminus \{F\}$, i. e. (A1) holds. Two distinct points $x, y \in \mathcal{P} \setminus F$ lie according to (P2) on exactly one line $G \in \mathcal{G}$. Because of $x, y \notin F$, x, y also lie on $G \setminus F$. Thus (A2) holds.

Now let $x \in \mathcal{P} \setminus F$ and $G \in \mathcal{G} \setminus \{F\}$ be given. Let $y := F \wedge G$ (according to (P3)) and $H := xy$. In the case $G \neq H$, $G \wedge H = y \in F$ and $(G \setminus F) \parallel (H \setminus F)$. For (A3) we must additionally show that $H \setminus F$ is the only parallel of $G \setminus F$ through x. So let $x \in (K \setminus F) \parallel (G \setminus F)$. In the case $K \setminus F = G \setminus F$, $K = G$ according to (P1) and (P2). Because of $x \in K = G$, then also $H = xy = G = K$. So let $K \setminus F$ and $G \setminus F$ be disjoint. Then it follows $G \wedge K \in F$, $y = F \wedge G \in K$ and $H = xy = K$. Thus the parallel axiom (A3) holds.

Finally, let $x, y, z \in \mathcal{P}$ be in general position (according to (P4)). In the case $x, y, z \in \mathcal{P} \setminus F$, x, y, z are also in general position with respect to $\mathcal{E}_F$. Wlog. let $z \in F$ and $x \notin F$. Since $xz \neq F$ has at least three points (according to (P1)), we can replace z by a point from $xz \setminus F$. In doing so, xz does not change. In particular, $y \notin xz$ still holds. With the same argument, one can assume $y \notin F$. Now $x, y, z \in \mathcal{P} \setminus F$ are in general position with respect to $\mathcal{E}_F$. Thus (A4) holds. ☐

Remark 6.7.

(i) With Theorem 6.6, Definition 1.11 and Theorem 1.12 can be transferred to projective planes: For every projective plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ there exists an $n \in \mathbb{N} \cup \{\infty\}$ with $|\mathcal{P}| = |\mathcal{G}| = n^2 + n + 1$ and every line contains exactly $n + 1$ points. One then also calls $\operatorname{ord} \mathcal{E} := n$ the *order* of $\mathcal{E}$.

(ii) If one starts with an affine plane $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ and chooses the line $F = [\mathcal{G}]$ in $\overline{\mathcal{E}}$, then obviously $\overline{\mathcal{E}}_F \cong \mathcal{E}$.

(iii) If $\varphi\colon \mathcal{E} \to \mathcal{E}'$ is an isomorphism of projective planes, then $\mathcal{E}_F \cong \mathcal{E}'_{\varphi(F)}$ holds, as one can easily verify. If $\mathrm{Aut}(\mathcal{E})$ operates transitively on the set of lines, then the isomorphism type of $\mathcal{E}_F$ does not depend on the choice of F. This holds, for example, for the projective coordinate plane $\overline{\mathcal{E}(K)}$ over a skew field K (see Example 6.5).

(iv) In general, however, the isomorphism type of $\mathcal{E}_F$ depends on the choice of F. For this, let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be the translation plane of order 9 over the quasifield Q from Theorem 4.18. Let $\overline{G_0} =$

$G_0 \cup [G_0]$ be the line in $\overline{\mathcal{E}}$ and $\mathcal{E}' := \overline{\mathcal{E}}_{\overline{G_0}}$. Because of $\mathcal{E} \cong \overline{\mathcal{E}}_{[\mathcal{G}]}$, it suffices to show that $\mathcal{E}'$ is not a translation plane. We denote the lines in $\mathcal{E}'$ by

$$\begin{aligned} G'_x &:= (G_x \cup [G_x]) \setminus \overline{G_0} = G_x = \{x\} \times Q \qquad (x \in \mathbb{F}_9^\times), \\ G'_{a,b} &:= G_{a,b} \setminus \{(0,b)\} \cup [G_{a,b}] \qquad (a, b \in Q), \\ [\mathcal{G}]' &:= [\mathcal{G}] \setminus \{[G_0]\}. \end{aligned}$$

The lines G'_x are obviously parallel to $[\mathcal{G}]'$. The lines $G_{a,b}$ and $G_{a',b'}$ are parallel if and only if $b = b'$. We apply the small theorem of Desargues to $F := G_{1,0}$, $G := G_{-1,0}$ and $H := G_{\zeta,0}$, where $\zeta \in \mathbb{F}_9$ with $\zeta^2 = -1$. Let $f := (1,1)$, $f' := (-1,-1)$, $g := (1,-1)$, $g' := (-1,1)$, $h = (-\zeta,1)$ and $h' := (1-\zeta, 1+\zeta)$. Obviously $fg = G'_1 \parallel G'_{-1} = f'g'$. Because of $N(\zeta) = \zeta^4 = 1$, $\zeta * x = \zeta x$ for all $x \in Q$. In particular, $h, h' \in H$. On the other hand, $(1+\zeta) * x = (1+\zeta)x^3$, because

$$N(1+\zeta) = (1+\zeta)(1+\zeta^3) = (1+\zeta)(1-\zeta) = -1.$$

This shows $gh = G'_{-1-\zeta,\zeta} \parallel G'_{\zeta-1,\zeta} = g'h'$. Finally, $fh = G'_{0,1} \nparallel G'_{-\zeta,-1-\zeta} = f'h'$. Thus $\mathcal{E}'$ is not a translation plane.

Theorem 6.8. *For every projective plane $\mathcal{E}$ and every line F of $\mathcal{E}$, $\mathcal{E} \cong \overline{\mathcal{E}_F}$ holds.*

Proof. Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$. Every point $x \in \overline{\mathcal{E}_F}$ lies either in $\mathcal{P} \setminus F$ or has the form $[G \setminus F]$ for $G \in \mathcal{G} \setminus \{F\}$. In the first case we define $\varphi(x) := x$ and otherwise $\varphi(x) := G \wedge F$. In the case $G \wedge F = H \wedge F$ we have $(G \setminus F) \parallel (H \setminus F)$, i. e. $[G \setminus F] = [H \setminus F]$. This shows the injectivity of φ. Given $x \in F$, choose $y \in \mathcal{P} \setminus F$ and set $G := xy \in \mathcal{G}$. Obviously $\varphi([G \setminus F]) = G \wedge F = x$ holds. Thus φ maps surjectively to $\mathcal{P}$.

Let $G' := (G \setminus F) \cup \{[G \setminus F]\}$ be a line in $\overline{\mathcal{E}_F}$. Obviously then $\varphi(G') = G \in \mathcal{G}$. Finally, let $G' := \{[G \setminus F] : G \neq F\}$ be the distinguished line in $\overline{\mathcal{E}_F}$. Then obviously $\varphi(G') = F$ holds. Consequently $\varphi\colon \overline{\mathcal{E}_F} \to \mathcal{E}$ is an isomorphism. □

Remark 6.9. Let $\mathcal{E}$ and $\mathcal{E}'$ be projective planes with lines F, F', such that $\mathcal{E}_F \cong \mathcal{E}'_{F'}$. From Remark 6.4 and Theorem 6.8 it then follows that $\mathcal{E} \cong \overline{\mathcal{E}_F} \cong \overline{\mathcal{E}'_{F'}} \cong \mathcal{E}'$.

Theorem 6.10 (Duality of projective planes)**.** *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be a projective plane and $\mathcal{P}^* := \{[x] : x \in \mathcal{P}\}$. Then $\mathcal{E}^* := (\mathcal{G}, \mathcal{P}^*)$ is a projective plane. One calls $\mathcal{E}^*$ the* dual *plane to $\mathcal{E}$. It holds that $(\mathcal{E}^*)^* \cong \mathcal{E}$.*

Proof. Theorem 1.12 applied to $\mathcal{E}_F$ shows $|[x]| \geq 3$ for $x \in \mathcal{P}$, so that (P1) holds for $\mathcal{E}^*$. Distinct lines $G, H \in \mathcal{G}$ lie only in $[G \wedge H]$, i. e. (P2) holds for $\mathcal{E}^*$. For distinct $x, y \in \mathcal{P}$ we have $[x] \wedge [y] = xy$. Thus (P3) is shown. Let $x, y, z \in \mathcal{P}$ be in general position. Then $xy, xz, yz \in \mathcal{G}$ are in general position with respect to $\mathcal{E}^*$, because otherwise $x = y = z$ would be the common intersection point of these lines. (P4) follows.

For the second statement we consider the bijection $\varphi\colon \mathcal{P} \to \mathcal{P}^*$, $x \mapsto [x]$. For $G \in \mathcal{G}$ we have $\varphi(G) = \{[x] : x \in G\} \in \mathcal{G}^*$. Therefore $\varphi\colon \mathcal{E} \to (\mathcal{E}^*)^*$ is an isomorphism. □

Definition 6.11. For a ring $(R, +, \cdot)$ let $R^\circ := (R, +, *)$ be the *opposite* ring with $x * y := y \cdot x$.

Theorem 6.12. *Let $\mathcal{E}$ be the projective coordinate plane over a skew field K. Then $\mathcal{E}^*$ is isomorphic to the projective coordinate plane over K°. In particular, $\mathcal{E}$ is* self-dual*, i. e. $\mathcal{E} \cong \mathcal{E}^*$, if K is a field.*

Proof. Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ and $\mathcal{E}^* = (\mathcal{G}, \mathcal{P}^*)$. For a subspace $W \in \mathcal{G}$ let

$$W^\perp := \{v \in K^3 : \forall w \in W : wv^{\mathrm{t}} = 0\}.$$

Because of $wv^TT = v * w^{\mathrm{t}}$, $W^\perp$ is a 1-dimensional K°-vector space, thus a point of the projective coordinate plane $\mathcal{E}^\circ = (\mathcal{P}^\circ, \mathcal{G}^\circ)$ over K°. Because of $W = \{w \in K^3 : \forall v \in W^\perp : wv^{\mathrm{t}} = 0\}$, the map $\varphi\colon \mathcal{G} \to \mathcal{E}^\circ$, $W \mapsto W^\perp$ is a bijection. For $V \in \mathcal{P}$ and $W \in [V] \in \mathcal{P}^*$ it holds that $W^\perp \subseteq V^\perp$. It follows that

$$\varphi([V]) = \{U \in \mathcal{P}^\circ : U \subseteq V^\perp\} \in \mathcal{G}^\circ.$$

Therefore $\varphi\colon \mathcal{E}^* \to \mathcal{E}^\circ$ is an isomorphism. If K is a field, then $K = K^\circ$ and $\mathcal{E}^* \cong \mathcal{E}^\circ$. □

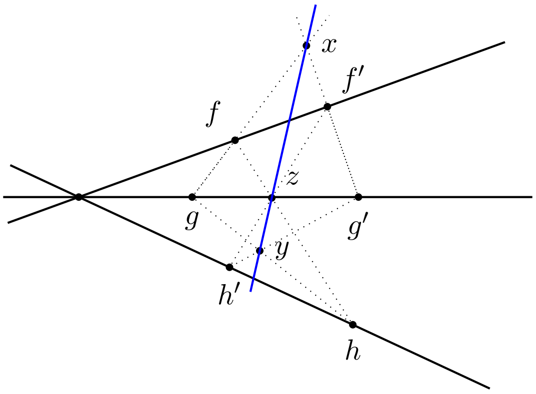
Example 6.13.

(i) Let K be a field whose Brauer group[2] contains an element $[A]$ of order > 2 (for example $K = \mathbb{Q}$). Here we can assume that A is a division algebra. The algebra A° then belongs to $[A]^{-1} \neq [A]$. In particular, $A \ncong A^\circ$. Suppose there exists an isomorphism φ between the projective coordinate planes $\mathcal{E}$ over A and $\mathcal{E}^\circ$ over A°, respectively. Then $\mathcal{E}_F \cong \mathcal{E}^\circ_{\varphi(F)}$ would hold, in contradiction to Theorem 4.10. Thus $\mathcal{E}$ is not self-dual.

(ii) The projective closure of the plane constructed in Theorem 4.18 is not self-dual, since the underlying quasifield satisfies only the left distributive law, but not the right distributive law (without proof).

Remark 6.14. By passing to the projective closure, statements about affine planes can be transferred to projective planes and additionally dualized there. Collinear points in $\mathcal{E}^*$ correspond to lines in $\mathcal{E}$ with a common intersection point.

Theorem 6.15 (Projective Desargues)**.** *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be the projective coordinate plane over a skew field K. Let $F, G, H \in \mathcal{G}$ be pairwise distinct and $f, f' \in F$, $g, g' \in G$, $h, h' \in H$ be pairwise distinct. F, G, H have a common intersection point if and only if $x := fg \wedge f'g'$, $y := gh \wedge g'h'$, $z := fh \wedge f'h'$ are collinear.*



Proof. Since F, G, H are pairwise distinct, x, y, z are well-defined. First, let us assume that F, G, H intersect in a point $s \in \mathcal{P}$. In the case $x = y$, x, y, z are collinear. So let $x \neq y$ and $L := xy \in \mathcal{G}$. In the case $f \in L$, then $L = fx = fg = gy = gh = fh \ni z$ and $x, y, z \in L$ are collinear (because $f \neq f'$, it follows $x \neq f$ and analogously $y \neq g$). We can therefore assume that none of the points f, f', g, g', h, h' lie on L.

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[2] See Algebra notes

According to Remark 6.7, $\mathcal{E}_L$ is isomorphic to the (affine) coordinate plane over K. The lines $F \setminus L$, $G \setminus L$, $H \setminus L$ in $\mathcal{E}_L$ are either parallel (if $s \in L$) or they have the common intersection point s. On the other hand, $(fg \setminus L) \parallel (f'g' \setminus L)$ as well as $(gh \setminus L) \parallel (g'h' \setminus L)$. From the (small) Theorem of Desargues, it follows that $(fh \setminus L) \parallel (f'h' \setminus L)$. This shows $z \in L$. Thus $x, y, z \in L$ are collinear.

Conversely, let $x, y, z \in L \in \mathcal{G}$ be collinear. In $\mathcal{E}^*$, then $[x]$, $[y]$, $[z]$ intersect in a point (namely L). By definition, $[x] = (fg)(f'g')$, $[y] = (gh)(g'h')$ and $[z] = (fh)(f'h')$. According to Theorem 6.12, we can apply the first part of the proof to the points fg, $f'g'$, gh, $g'h'$, fh, $f'h'$ of $\mathcal{E}^*$. It follows that

$$\begin{aligned}
&(fg)(gh) \wedge (f'g')(g'h') = [g] \wedge [g'] = gg', \\
&(gh)(fh) \wedge (g'h')(f'h') = hh', \\
&(fg)(fh) \wedge (f'g')(f'h') = ff'
\end{aligned}$$

are collinear, i.e., they have a common intersection point. □

Theorem 6.16 (Projective Pappus). *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be the projective coordinate plane over a field K. Let $F, G \in \mathcal{G}$ be distinct and $f, f', f'' \in F$, $g, g', g'' \in G$ be pairwise distinct. Then the intersection points $x := fg' \wedge f'g$, $y := f'g'' \wedge f''g'$, $z := f''g \wedge fg''$ are collinear.*

Proof. Because $F \neq G$, x, y, z are well-defined. We can assume $x \neq y$ and define $L := xy \in \mathcal{G}$. In the case $f \in L$, it would be

$$L = fx = fg' = yg' = f''g' = ff'' = F = xf' = f'g = gg' = G.$$

Therefore, L contains none of the points f, f', f'', g, g', g''. In the Pappus plane $\mathcal{E}_L$, it holds that $(fg' \setminus L) \parallel (f'g \setminus L)$ and $(f'g'' \setminus L) \parallel (f''g' \setminus L)$. It follows that $(fg'' \setminus L) \parallel (f''g \setminus L)$ and $z \in L$. □

Remark 6.17. Theorem 6.16 is exactly the statement of Pascal's Theorem 5.7 for projective planes. The advantage of the projective version is that the case distinction in the proof of Theorem 5.7 becomes redundant. The statement of Pappus, on the other hand, makes little sense for projective planes, since parallel lines are identical.

Theorem 6.18 (Brianchon). *Let $\mathcal{E} = (\mathcal{P}, \mathcal{G})$ be the projective coordinate plane over a field K. Let $f, g \in \mathcal{P}$ be distinct and $F, F', F'' \in [f]$, $G, G', G'' \in [g]$ be pairwise distinct. Then the lines $(F \wedge G')(F' \wedge G)$, $(F' \wedge G'')(F'' \wedge G')$ and $(F \wedge G'')(F'' \wedge G)$ have a common intersection point.*

Proof. The statement is equivalent to Theorem 6.16 for $\mathcal{E}^* \cong \mathcal{E}$ and is therefore correct. □

Remark 6.19. Brianchon's Theorem also holds for affine planes, provided that the used intersection points $F \wedge G'$ etc. exist.

7 Euclidean Geometry

Remark 7.1. In this section, we exclusively consider the Euclidean plane $\mathcal{E} = \mathcal{E}(\mathbb{R}) = (\mathcal{P}, \mathcal{G})$.

Definition 7.2.

(i) For $a, b \in \mathbb{R}^2$, let $\langle a, b\rangle := ab^{\mathrm{t}} \in \mathbb{R}$ be the *scalar product* as usual and $|a| := \sqrt{\langle a, a\rangle}$ the *norm*. In the case $\langle a, b\rangle = 0$, a and b are called *orthogonal* and one writes $a \perp b$. For $a = (x, y)$, one sets $a^{\perp} := (-y, x)$.

(ii) Lines $G, H \in \mathcal{G}$ are called *orthogonal* or *perpendicular*, if there exist distinct $g, g' \in G$ and $h, h' \in H$ with $g - g' \perp h - h'$. This obviously depends only on the direction of G and H. If applicable, one writes $G \perp H$.

(iii) Points $a, b, c \in \mathbb{R}^2$ in general position form a *triangle* with vertices a, b, c and *sides* ab, bc and ac. One calls $|a - b|$, $|b - c|$, $|a - c|$ the *lengths* of the sides, $|a - b| + |b - c| + |c - a|$ the *perimeter* and $\frac{1}{2}|[a, b, c]|$ the *area* of the triangle. If two sides have the same length, the triangle is called *isosceles*. If all sides have the same length, one speaks of *equilateral* triangles. If, on the other hand, two sides are orthogonal, the triangle is called *right-angled*.

Remark 7.3.

(i) Note that the terms “length” and “area” have no meaning as long as they are not associated with a measure (such as the Jordan measure). We initially dispense with the term “angle”.

(ii) The *line segment* between a and b consists of all points of the form $\alpha a + \beta b$ with $0 \le \alpha, \beta \le 1$ and $\alpha + \beta = 1$. The points in the *interior* of the triangle a, b, c therefore have the form $\alpha a + \beta b + \gamma c$ with $0 \le \alpha, \beta, \gamma \le 1$ and $\alpha + \beta + \gamma = 1$. Two points $a, b \in \mathcal{P}$ lie on the *same side* of a line G if the line segment between a and b is disjoint from G.

(iii) The map $\mathcal{P} \to \mathcal{P}$, $a \mapsto a^{\perp}$ is obviously linear (it corresponds to a rotation by 90° counterclockwise). Furthermore,

$$(a^{\perp})^{\perp} = -a, \qquad \langle a, a^{\perp}\rangle = 0, \qquad |a^{\perp}| = |a|, \qquad \langle a^{\perp}, b^{\perp}\rangle = \langle a, b\rangle, \qquad \langle a^{\perp}, b\rangle = [a, b].$$

Lemma 7.4. *For $a, b \in \mathbb{R}^2$ we have*

$$\langle a, b\rangle^2 \le \langle a, b\rangle^2 + [a, b]^2 = |a|^2|b|^2 \qquad (\textit{Cauchy-Schwarz inequality})$$

$$||a| - |b|| \le |a + b| \le |a| + |b| \qquad (\textit{triangle inequality}).$$

Equality $|a + b| = |a| + |b|$ holds if and only if $a = \lambda b$ for some $\lambda \ge 0$.

Proof. With $a = (a_x, a_y)$ and $b = (b_x, b_y)$ we have

$$\langle a, b\rangle^2 + [a, b]^2 = (a_x b_x + a_y b_y)^2 + (a_x b_y - a_y b_x)^2 = (a_x^2 + a_y^2)(b_x^2 + b_y^2) = |a|^2|b|^2.$$

From this it follows that

$$|a + b|^2 = \langle a + b, a + b\rangle = \langle a, a\rangle + 2\langle a, b\rangle + \langle b, b\rangle \le |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2$$

with equality if and only if $\langle a, b\rangle = |a||b|$, hence $[a, b] = 0$. This shows $a = \lambda b$ with $\lambda \ge 0$.

For the left-hand estimate we use $|a| = |a + b - b| \le |a + b| + |b|$ and $|a| - |b| \le |a + b|$. Swapping a and b yields $-(|a| - |b|) = |b| - |a| \le |a + b|$, hence $||a| - |b|| \le |a + b|$. ☐

Remark 7.5. The equation

$$\frac{\langle a,b\rangle^2}{|a|^2|b|^2}+\frac{[a,b]^2}{|a|^2|b|^2}=1$$

is a version of the *trigonometric Pythagoras*.

Lemma 7.6. *For $G \in \mathcal{G}$ there exists, up to parallelism, exactly one line $H \in \mathcal{G}$ with $G \perp H$. One writes $[G]^\perp := [H]$. For $a \in \mathcal{P}$ there exists exactly one $H \in \mathcal{G}$ with $a \in H$ and $H \perp G$. One calls $d(a,G) := |a - G \wedge H|$ the* distance *from a to G. For distinct $g, g' \in G$ we have*

$$G \wedge H = g + \frac{\langle a-g, g-g'\rangle}{|g-g'|^2}(g-g'), \qquad d(a,G) = \frac{|[a-g,g-g']|}{|g-g'|}.$$

Proof. Let $v = (x,y) \in \mathcal{P} \setminus \{0\}$ be the direction of G. The lines orthogonal to G then have the direction $(-y,x)$, so they are parallel to each other. The second assertion follows from the parallel axiom. For the last statement, we can assume $g - g' = v$ and $|v| = 1$. The point $b := g + \langle a-g, v\rangle v$ obviously lies on G. Because of

$$\langle a-b,v\rangle = \langle a-g,v\rangle - \langle a-g,v\rangle = 0$$

b also lies on H. Let $a = (a_x, a_y)$ and $g = (g_x, g_y)$. According to Lemma 7.4, it holds that

$$d(a,G)^2 = |a-b|^2 = |a-b|^2|v|^2 = [a-b,v]^2 = [a-g+g-b,v]^2 = [a-g,v]^2. \qquad \square$$

Definition 7.7.

(i) Let a, b, c be a triangle. The line orthogonal to ab through c (resp. $\frac{1}{2}(a+b)$) is called the *altitude* (resp. *perpendicular bisector*) of ab. The line $\frac{1}{2}(a+b)c$ is called the *median* of ab.

(ii) For $m \in \mathcal{P}$ and $\rho > 0$ let

$$K_\rho(m) := \{a \in \mathcal{P} : |a-m| = \rho\} \subseteq \mathcal{P}$$

be the *circle* with *center* m and *radius* ρ.

Theorem 7.8. *The medians of a triangle a, b, c intersect at the* centroid $\frac{1}{3}(a+b+c)$. *The altitudes intersect at the* orthocenter.

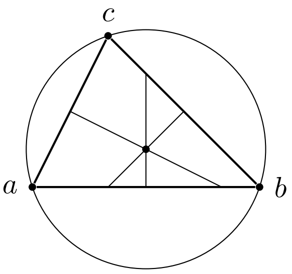
Proof. The first assertion follows from Theorem 5.17. Since ab and bc are not parallel, the altitudes H_{ab} and H_{bc} intersect at a point h. Here it holds that

$$\langle h,a\rangle - \langle h,b\rangle - \langle c,a-b\rangle = \langle h-c,a-b\rangle = 0 = \langle h-a,c-b\rangle = \langle h,c\rangle - \langle h,b\rangle - \langle a,c-b\rangle$$

and it follows that $\langle h-b,a-c\rangle = \langle h,a\rangle - \langle h,c\rangle - \langle b,a-c\rangle = 0$. Thus h also lies on H_{ac}. $\square$

Theorem 7.9. *The vertices of a triangle a, b, c lie on exactly one circle, which is called the* circumcircle *of a, b, c. The circumcenter is the common intersection point of the perpendicular bisectors. The radius of the circumcircle is*

$$\frac{|a-b||b-c||c-a|}{2|[a,b,c]|}.$$



Proof. Let M_{ab} be the perpendicular bisector of ab. Then

$$\begin{aligned} v \in M_{ab} &\iff 0 = \langle a-b, v - \frac{1}{2}(a+b)\rangle = \langle a, v\rangle - \langle b, v\rangle - \frac{1}{2}(|a|^2 - |b|^2) \\ &\iff |v-a|^2 = |v|^2 - 2\langle v, a\rangle + |a|^2 = |v|^2 - 2\langle v, b\rangle + |b|^2 = |v-b|^2. \end{aligned}$$

Since ab and bc are not parallel, M_{ab} and M_{bc} must intersect in a point m. Because of $r := |m-a| = |m-b| = |m-c|$, m also lies on M_{ac}. Furthermore, a, b, c lie on $K_r(m)$. If a, b, c also lie on $K_{r'}(m')$, then $m' = M_{ab} \wedge M_{bc} = m$ and $r = |v-m| = |v-m'| = r'$. For the last statement, we can assume $c = (0,0)$ after translation. Because of $[a,b,c] = [a,b] = \langle a^\perp, b\rangle$, it then follows that

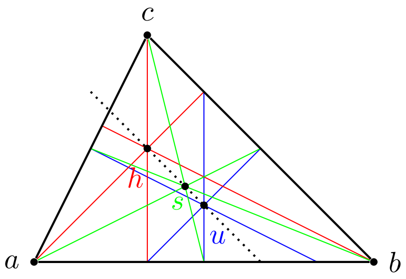
$$m = \frac{1}{2[a,b]}(|b|^2 a^\perp - |a|^2 b^\perp).$$

The square of the radius is

$$|m|^2 = \frac{1}{4[a,b]^2}(|a|^2|b|^4 + |a|^4|b|^2 - 2|a|^2|b|^2\langle a, b\rangle) = \Big(\frac{|a||b||a-b|}{2|[a,b]|}\Big)^2. \qquad \square$$

Corollary 7.10. *Two distinct circles intersect in at most two points.*

Theorem 7.11 (Euler)**.** *For the circumcenter u, the centroid s, and the orthocenter h of a triangle, the following holds:* $\boxed{3s = 2u + h.}$ *In particular, all three points lie on a line.*



Proof. By translation by the vector v, the right and left sides of the equation are added to $3v$. We can therefore assume $u = 0$. Then $|a| = |b| = |c|$ and

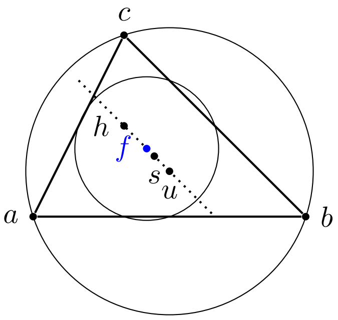
$$\langle 3s - c, a - b\rangle = \langle a + b, a - b\rangle = 0 = \langle 3s - b, a - c\rangle = \langle 3s - a, b - c\rangle.$$

Thus $3s = h$. The second assertion follows from Lemma 5.11. $\square$

Remark 7.12. If $u = h$ in Theorem 7.11, then c lies on the perpendicular bisector M_{ab} and the proof of Theorem 7.9 shows $|c-a| = |c-b|$. Consequently, a, b, c is then equilateral. In all other cases, the line in Theorem 7.11 is uniquely determined and is called the *Euler line* of a, b, c.

Definition 7.13. The circle through the midpoints of the sides of a triangle is called the *Feuerbach circle*. Its center f is called the *Feuerbach point*.

Theorem 7.14 (Feuerbach)**.** *For the circumcenter u, the centroid s and the Feuerbach point f of a triangle,* $\boxed{3s = u + 2f.}$ *holds. In particular, f lies on the Euler line (if not equilateral) exactly in the middle of u and the orthocenter. The radius of the Feuerbach circle is half as large as the circumradius.*



Proof. This time we can assume $s = 0$ by translation. The midpoints of the sides then have the form $-\frac{1}{2}a$, $-\frac{1}{2}b$ and $-\frac{1}{2}c$. Because $|2f + a| = |2f + b| = |2f + c|$, $-2f$ satisfies exactly the equations of u. Thus $3s = u + 2f$ is proven and the radius of the Feuerbach circle is half as large as the radius of the circumcircle. From Theorem 7.11 one obtains $2u + h = u + 2f$, so $f = \frac{1}{2}(u + h)$. ☐

Theorem 7.15. *Let a triangle a, b, c be given with altitude foot h_c on ab and side lengths $A = |b - c|$, $B = |a - c|$, $C = |a - b|$, $H = |c - h_c|$, $P = |b - h_c|$, $Q = |a - h_c|$. Then the following statements are equivalent:*

(1) *a, b, c is right-angled with $ac \perp bc$.*

(2) $A^2 + B^2 = C^2$ (Pythagoras).

(3) *The circumcenter lies on ab* (Thales).

(4) $H^2 = PQ$ (altitude theorem).

(5) $A^2 = QC$ *and* $B^2 = PC$ (cathetus theorem).

(6) $\frac{1}{A^2} + \frac{1}{B^2} = \frac{1}{H^2}$ (inverse Pythagoras).

Proof. (1)⇔(2):

$$C^2 = \langle a - b, a - b\rangle = \langle a - c + c - b, a - c + c - b\rangle = B^2 + 2\langle a - c, c - b\rangle + A^2.$$

(1)⇒(3): For the midpoint $m = \frac{1}{2}(a + b)$ of the side ab it holds that

$$|m - c|^2 = \left|\frac{1}{2}(a - c) + \frac{1}{2}(b - c)\right|^2 = \frac{1}{4}B^2 + \frac{1}{4}A^2 = \frac{1}{4}C^2 = \left|\frac{1}{2}(a - b)\right|^2 = |m - b|^2.$$

Therefore m is the circumcenter.

(3)⇒(4): Let $m = \frac{1}{2}(a + b)$. According to (3) it holds wlog. $m - a = \lambda(m - h_c)$ for a $\lambda > 0$ (swap a and b if necessary). We apply Pythagoras to the right-angled triangles m, h_c, c and b, h_c, c:

$$H^2 = |m - c|^2 - |m - h_c|^2 = |m - a|^2 - |m - h_c|^2 = (|m - a| + |m - h_c|)(|m - b| - |m - h_c|) \overset{7.4}{=} QP.$$

(4)⇒(5): Pythagoras on b, c, h_c and a, c, h_c yields

$$\begin{aligned} PC &= P(P+Q) = P^2 + H^2 = A^2, \\ QC &= Q(P+Q) = H^2 + Q^2 = B^2. \end{aligned}$$

(5)⇒(2):

$$A^2 + B^2 = C(P+Q) = C^2.$$

(4)⇔(6): It holds that

$$\begin{aligned} \frac{1}{A^2} + \frac{1}{B^2} = \frac{1}{H^2} &\iff H^2(A^2+B^2) = A^2B^2 \\ &\iff H^2(H^2+P^2+H^2+Q^2) = (H^2+P^2)(H^2+Q^2) \\ &\iff H^4 = P^2Q^2 \iff H^2 = PQ. \end{aligned}$$ □

Theorem 7.16. *With the notation from Theorem 7.15 it holds that*

$$\begin{aligned} A^2 + B^2 &= C^2 + 2\langle c-a, c-b\rangle && \text{(law of cosines)} \\ A|a-h_a| &= B|b-h_b| = C|c-h_c| && \text{(law of sines)} \end{aligned}$$

Proof. The law of cosines follows from

$$A^2 + B^2 = \langle b-c, b-c\rangle + \langle a-c, a-c\rangle = \langle a-b, a-b\rangle + 2\langle c-a, c-b\rangle.$$

For the law of sines we choose $\lambda \in \mathbb{R}$ with $h_a - c = \lambda(b-c)$. According to Lemma 7.4 it holds that

$$A|a-h_a| = |b-c||a-h_a| = |[b-c, a-h_a]| = |[b-c, a-h_a+\lambda(b-c)]| = |[a,b,c]|$$

and the claim follows for reasons of symmetry. □

Remark 7.17. The proof of Theorem 7.16 yields the well-known formula for the area $\frac{1}{2}|[a,b,c]| = \frac{1}{2}A|a-h_a|$. The next theorem provides a formula for the square of the area.

Theorem 7.18 (Heron's Formula)**.** *For a triangle with side lengths A, B, C, let $S := \frac{1}{2}(A+B+C)$ be the semiperimeter. Then*

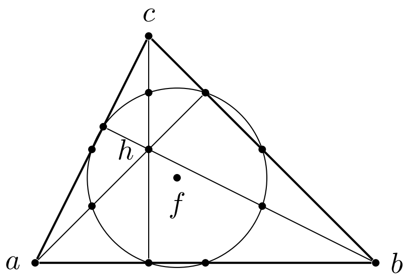
$$\frac{1}{4}[a,b,c]^2 = S(S-A)(S-B)(S-C).$$

Proof.

$$\begin{aligned} 16S(S-A)(S-B)(S-C) &= (A+B+C)(B+C-A)(A+C-B)(A+B-C) \\ &= ((A+B)^2 - C^2)(C^2 - (A-B)^2) \\ &= 4A^2B^2 - (A^2+B^2-C^2)^2 \\ &\overset{7.4+7.16}{=} 4\langle a-c, b-c\rangle^2 + 4[a-c, b-c]^2 - 4\langle c-a, c-b\rangle^2 \\ &= 4[a,b,c]^2 \end{aligned}$$ □

Theorem 7.19 (9-Point Theorem)**.** *The following points lie on the Feuerbach circle of a (non-equilateral) triangle* a, b, c*:*

- *The midpoints of the sides* $\frac{1}{2}(a+b)$, $\frac{1}{2}(a+c)$, $\frac{1}{2}(b+c)$.
- *The feet of the altitudes* h_a, h_b, h_c.
- *The midpoints* $\frac{1}{2}(h+a)$, $\frac{1}{2}(h+b)$, $\frac{1}{2}(h+c)$.



Proof. By definition, the midpoints of the sides lie on the Feuerbach circle. From $3s = 2u + h$ and $6s = 4f + 2u$ it follows that $4f = 3s + h = a + b + c + h$. Rearranging yields $f - \frac{1}{2}(a+b) = \frac{1}{2}(h+c) - f$ and $|f - \frac{1}{2}(a+b)| = |f - \frac{1}{2}(h+c)|$. Thus $\frac{1}{2}(h+c)$ lies on the Feuerbach circle and analogously also $\frac{1}{2}(h+a)$ and $\frac{1}{2}(h+b)$.

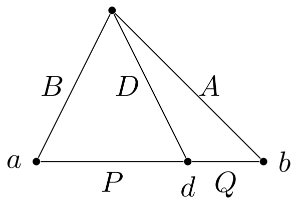
According to Theorem 7.14, f lies exactly between u and h. Let F and G be the parallels to the altitude ah_a through u and f respectively. According to the intercept theorem, $g := G \wedge bc$ then lies exactly between $\frac{1}{2}(b+c)$ and h_a. Pythagoras shows

$$|f - h_a|^2 = |f - g|^2 + |g - h_a|^2 = |f - g|^2 + |g - \frac{1}{2}(b+c)|^2 = |f - \frac{1}{2}(b+c)|.$$

Thus h_a lies on the Feuerbach circle and analogously h_b, h_c. ☐

Theorem 7.20 (Stewart)**.** *Let* a, b, c *be a triangle and* d *a point on the line segment between* a *and* b*. With the notations* $A := |b-c|$, $B := |a-c|$, $C := |a-b|$, $P := |d-a|$, $Q := |d-b|$ *and* $D := |c-d|$*, the following holds*

$$PA^2 + QB^2 = C(D^2 + PQ).$$



Proof. Using the law of cosines and $C = P + Q$, it holds that

$$\begin{aligned} PA^2 + QB^2 &= P(Q^2 + D^2 + 2\langle d-b, d-c\rangle) + Q(P^2 + D^2 + 2\langle d-c, d-a\rangle) \\ &= C(D^2 + PQ) + 2\langle d-c, P(d-b) + Q(d-a)\rangle. \end{aligned}$$

Since d lies between a and b, $d - a$ and $d - b$ are linearly dependent with opposite directions. From $|P(d-b)| = PQ = |Q(d-a)|$ it follows that $P(d-b) + Q(d-a) = 0$. This shows the claim. ☐

Definition 7.21. For a triangle a, b, c, one calls

$$\begin{aligned} W_a &:= a + \mathbb{R}(|c-a|(b-a) + |a-b|(c-a)), \\ W_b &:= b + \mathbb{R}(|a-b|(c-b) + |b-c|(a-b)), \\ W_c &:= c + \mathbb{R}(|b-c|(a-c) + |c-a|(b-c)) \end{aligned}$$

the (*inner*) *angle bisectors* and

$$\begin{aligned} W_a^* &:= a + \mathbb{R}(|c-a|(b-a) - |a-b|(c-a)), \\ W_b^* &:= b + \mathbb{R}(|b-c|(a-b) - |a-b|(c-b)), \\ W_c^* &:= c + \mathbb{R}(|c-a|(b-c) - |b-c|(a-c)) \end{aligned}$$

the *outer angle bisectors*.

Remark 7.22.

(i) Because of

$$\begin{aligned} &\langle |c-a|(b-a) + |a-b|(c-a), |c-a|(b-a) - |a-b|(c-a) \rangle \\ &= |c-a|^2|b-a|^2 - |a-b|^2|c-a|^2 = 0 \end{aligned}$$

$W_a \perp W_a^*$ and analogously $W_b \perp W_b^*$, $W_c \perp W_c^*$.

(ii) Let $a + \lambda v$ be an intersection point of the line $a + \mathbb{R}v$ with the circle $K_r(m)$. Then

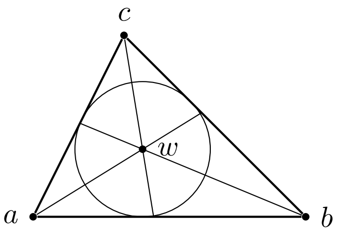
$$|a-m|^2 - 2\lambda\langle a-m, v\rangle + \lambda^2|v|^2 = |a + \lambda v - m|^2 = r^2.$$

Since this quadratic equation in λ has at most two solutions, the line and circle intersect in at most two points. If there is exactly one intersection point, then the line and circle *touch* each other. In this case, the line is called the *tangent* at the point of contact of the circle. We also say that two circles *touch* each other if they have exactly one point in common.

Theorem 7.23. *The internal angle bisectors of a triangle a, b, c intersect at the point*

$$w := \frac{1}{\sigma}(|b-c|a + |c-a|b + |a-b|c),$$

where $\sigma := |a-b| + |b-c| + |c-a|$ is the perimeter. The circle with center w and radius $|[a, b, c]|/\sigma$ touches the three sides of the triangle and is therefore called the incircle.



Proof. Because of

$$\begin{aligned}
w &= a + \frac{1}{\sigma}\big(|c-a|(b-a) + |a-b|(c-a)\big)\\
&= b + \frac{1}{\sigma}\big(|a-b|(c-b) + |b-c|(a-b)\big)\\
&= c + \frac{1}{\sigma}\big(|b-c|(a-c) + |c-a|(b-c)\big)
\end{aligned}$$

w lies in $W_a \cap W_b \cap W_c$. If two of the angle bisectors were parallel, they would already have to be equal. But then a, b, c would be collinear. Thus $w = W_a \wedge W_b = W_a \wedge W_c$.

For the second statement, we use Lemma 7.6. It holds that

$$d(w, ab) = \frac{|[w-a, a-b]|}{|a-b|} = \frac{|[c-a, a-b]|}{\sigma} = \frac{|[a,b,c]|}{\sigma} = d(w, bc) = d(w, ac).$$

By Pythagoras, the incircle intersects the sides of the triangle in exactly one point. □

Theorem 7.24. *The angle bisectors W_a, W_b^* and W_c^* intersect at*

$$w_a^* := \frac{1}{\sigma_a}(|c-a|b + |a-b|c - |b-c|a) \qquad \sigma_a := |c-a| + |a-b| - |b-c|.$$

This is the center of the excircle *with radius $\frac{|[a,b,c]|}{\sigma_a}$, which touches the three sides of the triangle.*

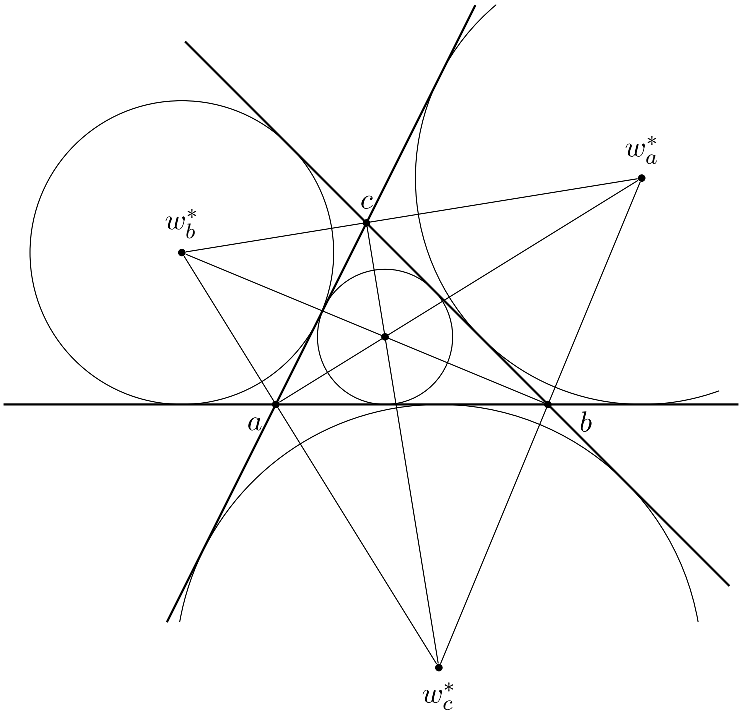
Proof. It holds that

$$\begin{aligned}
w_a^* &= a + \frac{1}{\sigma_a}(|c-a|(b-a) + |b-a|(c-a))\\
&= b + \frac{1}{\sigma_a}(|a-b|(c-b) - |c-b|(a-b))\\
&= c + \frac{1}{\sigma_a}(|a-c|(b-c) - |c-b|(a-c))
\end{aligned}$$

and

$$d(w_a^*, ab) = \frac{|[w_a^*-a, a-b]|}{|a-b|} = \frac{|[c-a, a-b]|}{\sigma_a} = \frac{|[a,b,c]|}{\sigma_a} = d(w_a^*, bc) = d(w_a^*, ac).$$ □

Remark 7.25. Of course, there are two further excircles at ab and ac.



Theorem 7.26 (Angle Bisector Theorem)**.** *In a triangle* a, b, c *let* $w_a := W_a \wedge bc$, $w_b := W_b \wedge ac$, $w_c := W_c \wedge ab$. *Then* $|a-b||c-w_a| = |b-c||b-w_a|$ *holds, i. e. the angle bisectors divide the opposite side in the ratio of the adjacent sides. In particular,*

$$|a-w_b||b-w_c||c-w_a| = |a-w_c||b-w_a||c-w_b|.$$

Proof. As usual, let $A := |b-c|$, $B := |a-c|$ and $C := |a-b|$. One easily sees:

$$w_c = c + \frac{1}{A+B}(A(a-c)+B(b-c)) = \frac{A}{A+B}a + \frac{B}{A+B}b.$$

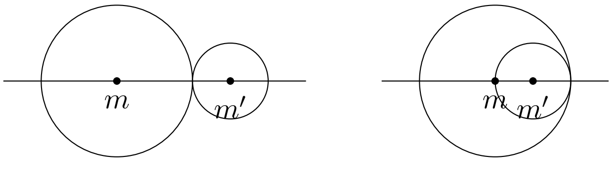
It follows

$$A|w_c - a| = \frac{AB}{A+B}|b-a| = A|w_c - b|.$$

By cyclic permutation one obtains

$$|a-w_b||b-w_c||c-w_a| = \frac{B}{A}|a-w_c| \cdot \frac{C}{B}|b-w_a| \cdot \frac{A}{C}|c-w_b| = |a-w_c||b-w_a||c-w_b|. \qquad \square$$

Lemma 7.27. *Distinct circles* $K_\rho(m)$ *and* $K_{\rho'}(m')$ *touch each other if and only if* $|m-m'| = \rho + \rho'$ *or* $|m-m'| = |\rho - \rho'|$. *If applicable, the point of tangency lies on* mm'.

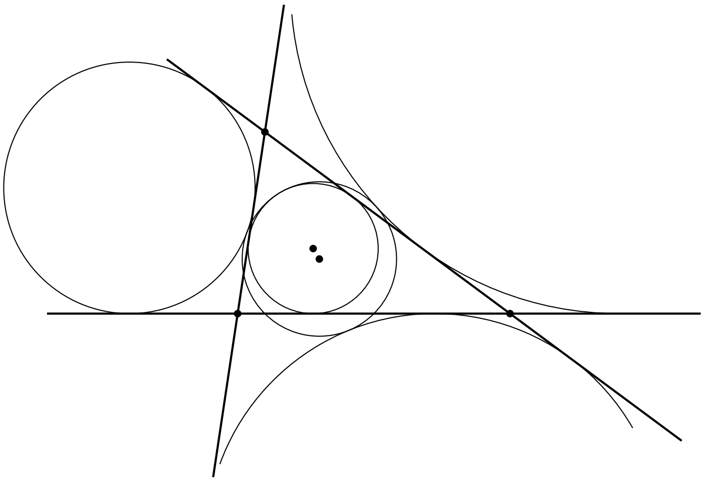


Proof. Wlog. let $m' = (0,0) \neq m$. Every $x \in \mathcal{P}$ can be uniquely written in the form $x = \lambda m + \mu m^{\perp}$. It holds $x \in K_{\rho}(m) \cap K_{\rho'}(0)$ if and only if

$$(\lambda^2 + \mu^2)|m|^2 = |x|^2 = (\rho')^2 \qquad ((\lambda-1)^2 + \mu^2)|m|^2 = |x-m|^2 = \rho^2.$$

If applicable, $(2\lambda - 1)|m|^2 = (\rho')^2 - \rho^2$ holds. This uniquely determines λ. The intersection point is unique if and only if the equations hold with $\mu = 0$. If applicable, $x = \lambda m \in mm'$ and $|m| = |x| + |m-x| = \rho' + \rho$ or $|m| = ||x| - |m-x|| = |\rho - \rho'|$ by the triangle inequality. □

Theorem 7.28 (FEUERBACH). *The Feuerbach circle of a triangle is tangent to the incircle and the three excircles.*



Proof. As usual, let $A := |b-c|$, $B := |a-c|$ and $C := |a-b|$. Wlog. let $(0,0)$ be the center of the circumcircle and $|a| = |b| = |c| = \frac{ABC}{2|[a,b,c]|}$ its radius (Theorem 7.9). According to Theorem 7.14, $f = \frac{1}{2}(a+b+c)$ is the center and $|a|/2$ the radius of the Feuerbach circle. Let w be the center and $|[a,b,c]|/\sigma$ the radius of the incircle (Theorem 7.23).

According to Heron's formula,

$$\begin{aligned} 4[a,b,c]^2 &= \sigma(A+B-C)(B+C-A)(C+A-B) = ((A+B)^2 - C^2)(C^2 - (A-B)^2) \\ &= 2A^2B^2 + 2B^2C^2 + 2C^2A^2 - A^4 - B^4 - C^4 \end{aligned}$$

On the other hand,

$$4\sigma^2|w-f|^2 = \left|(A-B-C)a + (B-A-C)b + (C-A-B)c\right|^2.$$

Due to $2\langle a,b\rangle = 2|a|^2 - \langle a-b, a-b\rangle = 2|a|^2 - C^2$, it follows that

$$\begin{aligned} 4\sigma^2|w-f|^2 = |a|^2\sigma^2 - \big(&A^2(B-A-C)(C-A-B) + B^2(A-B-C)(C-A-B) \\ &+ C^2(A-B-C)(B-A-C)\big) \\ = |a|^2\sigma^2 &+ A^2((C-B)^2 - A^2) + B^2((A-C)^2 - B^2) + C^2((A-B)^2 - C^2) \\ = \sigma^2|a|^2 &- 2\sigma ABC + 2A^2B^2 + 2B^2C^2 + 2C^2A^2 - A^4 - B^4 - C^4 \\ = (\sigma|a| &- 2|[a,b,c]|)^2. \end{aligned}$$

Therefore $|w-f|^2 = (\frac{|a|}{2} - \frac{|[a,b,c]|}{\sigma})^2$ holds and the first statement follows from Lemma 7.27.

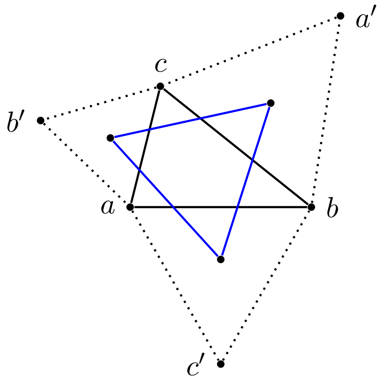
Now let w_a^* be the center and $\frac{|[a,b,c]|}{\sigma_a}$ the radius of the excircle as in Theorem 7.24. Then

$$\begin{aligned}4\sigma_a^2|w_a - f|^2 &= |-\sigma a + (A+B-C)b + (A+C-B)c|^2 = (-\sigma_a)^2|a|^2 - A^2(A+B-C)(A+C-B)\\&\quad + B^2\sigma(A+C-B) + C^2\sigma(A+B-C)\\&= \sigma_a^2|a|^2 + A^2((B-C)^2 - A^2) + B^2((A+C)^2 - B^2) + C^2((A+B)^2 - C^2)\\&= \sigma_a^2|a|^2 + 2\sigma_a ABC + 2A^2B^2 + 2B^2C^2 + 2C^2A^2 - A^4 - B^4 - C^4\\&= (\sigma_a|a| + 2|[a,b,c]|)^2.\end{aligned}$$

This shows $|w_a - f| = \frac{|a|}{2} + \frac{|[a,b,c]|}{\sigma_a}$ and the second statement follows. □

Remark 7.29. Since the Feuerbach circle touches or intersects all sides of the triangle, its radius is at least as large as the incircle radius. Therefore, $|w-f| + \frac{|[a,b,c]|}{\sigma} = \frac{ABC}{2|[a,b,c]|}$ holds.

Theorem 7.30 (NAPOLEON)**.** *Let a, b, c be a triangle and $a', b', c' \in \mathcal{P}$, such that the triangles $\Delta_c := \{a, b, c'\}$, $\Delta_a := \{b, c, a'\}$ and $\Delta_b := \{c, a, b'\}$ are equilateral and directed "outwards", i. e. a and a' lie on different sides of bc (analogously for b', c'). The centroids of Δ_a, Δ_b and Δ_c then form an equilateral triangle.*



Proof. Let $m_a := \frac{1}{2}(b+c)$ and s_a be the center of gravity of Δ_a. By Pythagoras it holds that

$$|m_a - a'|^2 = |a' - b|^2 - \frac{|b-c|^2}{4} = \frac{3}{4}|b-c|^2.$$

Since s_a is also the circumcenter of Δ_a, it holds that

$$\begin{aligned}\frac{|b-c|^2}{4} &= |b - s_a|^2 - |m_a - s_a|^2 = (|a' - m_a| - |m_a - s_a|)^2 - |m_a - s_a|^2\\&= \frac{3}{4}|b-c|^2 - \sqrt{3}|b-c||m_a - s_a|.\end{aligned}$$

It follows that $|m_a - s_a| = \frac{1}{2\sqrt{3}}|b-c|$. Because $|b-c| = |(b-c)^\perp|$, s_a can now be calculated exactly:

$$s_a = m_a + \frac{1}{2\sqrt{3}}(b-c)^\perp.$$

With Remark 7.3 we obtain

$$\begin{aligned}4|s_a - s_b|^2 &= |b - a + \frac{1}{\sqrt{3}}(a + b - 2c)^\perp|^2 \\ &= |b - a|^2 + \frac{2}{\sqrt{3}}\langle b - a, (b - a + 2(a - c))^\perp\rangle + \frac{1}{3}|a + b - 2c|^2 \\ &= |b - a|^2 + \frac{4}{\sqrt{3}}\langle b - a, (a - c)^\perp\rangle + \frac{1}{3}|a + b|^2 - \frac{4}{3}\langle a + b, c\rangle + \frac{4}{3}|c|^2 \\ &= \frac{4}{3}(|a|^2 + |b|^2 + |c|^2 - \langle a, b\rangle - \langle a, c\rangle - \langle b, c\rangle) + \frac{4}{\sqrt{3}}[b - a, c - a] \\ &= \frac{4}{3}(|a|^2 + |b|^2 + |c|^2 - \langle a, b\rangle - \langle a, c\rangle - \langle b, c\rangle) + \frac{4}{\sqrt{3}}[a, b, c]\end{aligned}$$

Since this expression is invariant under cyclic permutation of a, b, c, the sides of s_a, s_b, s_c have the same length. □

Remark 7.31. Napoleon's theorem also holds if one directs the triangles $\Delta_a, \Delta_b, \Delta_c$ "inwards" and assumes that a, b, c is not equilateral (otherwise the centers of gravity of $\Delta_a, \Delta_b, \Delta_c$ coincide). In the proof, only the sign of $\frac{4}{\sqrt{3}}[a, b, c]$ changes.

Remark 7.32. According to Theorem 3.9, any two triangles can be transformed into each other by an affine isomorphism $f = (v, A) \in \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$. If A is orthogonal (i.e., $A^{\mathrm{t}}A = 1_2$), then the triangles are called *congruent*. In the case $A^{\mathrm{t}}A = \lambda 1_2$ with $\lambda \in \mathbb{R}$, the triangles are called *similar*.

Definition 7.33. For a triangle a, b, c we denote the numbers

$$\alpha := \frac{\langle a - b, a - c\rangle}{|a - b||a - c|}, \qquad \beta := \frac{\langle b - a, b - c\rangle}{|b - a||b - c|}, \qquad \gamma := \frac{\langle c - a, c - b\rangle}{|c - a||c - b|}$$

as *angles*.

Remark 7.34.

(i) Our angles are actually the cosine values of the "angles". Since the cosine is injective on $[0, \pi]$, the viewpoints are equivalent. By the Cauchy-Schwarz inequality, $-1 < \alpha, \beta, \gamma < 1$ holds.

(ii) A triangle is right-angled if and only if one angle is 0. It is equilateral if and only if all angles are $\frac{1}{2}$. If all angles are positive (or one angle is negative), the triangle is called *acute-angled* (or *obtuse-angled*).

(iii) The next theorem corresponds to the statement that the sum of interior angles in a triangle is always π.

Lemma 7.35 ("Sum of interior angles")**.** *For the angles α, β, γ of a triangle, it holds that*

$$\gamma = \sqrt{(1 - \alpha^2)(1 - \beta^2)} - \alpha\beta.$$

Proof. By Lemma 7.4 we have

$$\begin{aligned}\sqrt{(1-\alpha^2)(1-\beta^2)}-\alpha\beta &= \frac{|[a-b,a-c]|}{BC}\frac{|[b-a,b-c]|}{AC}-\frac{\langle a-b,a-c\rangle\langle b-a,b-c\rangle}{ABC^2}\\ &= \frac{[a-b,a-c]^2-\langle a-b,a-c\rangle\langle b-a,b-c\rangle}{ABC^2}\\ &= \frac{B^2C^2-\langle a-b,a-c\rangle^2-\langle a-b,a-c\rangle\langle b-a,b-c\rangle}{ABC^2}\\ &= \frac{B^2C^2-\langle a-b,a-c\rangle C^2}{ABC^2}=\frac{\langle a-c,a-c\rangle-\langle a-b,a-c\rangle}{AB}\\ &= \frac{\langle c-a,c-b\rangle}{AB}=\gamma.\end{aligned}$$

□

Theorem 7.36 (Congruence theorems)**.** *For triangles Δ and Δ', the following statements are equivalent:*

(K) *Δ and Δ' are congruent.*

(SSS) *The side lengths of Δ and Γ coincide (up to their order).*

(SWS) *Two side lengths and the included angle coincide.*

(SSW) *Two side lengths and the angle opposite the larger side coincide.*

(SWW) *One side length and two angles coincide.*

Proof. Let a, b, c be the vertices, A, B, C the side lengths and α, β, γ the angles of Δ. Let the corresponding quantities for Δ' be a', b', c' etc.

(K)⇒(SSS): Let $f = (v, S) \in \mathrm{Aut}(\mathcal{E})$ with $S^{\mathrm{t}}S = 1_2$ and $f(\Delta) = \Delta'$. Then $C' = |a' - b'| = |Sa + v - Sb - v| = |a - b| = C$ etc. holds.

(SSS)⇒(SWS): By the law of cosines, it holds that

$$\alpha' := \frac{\langle a'-b',a'-c'\rangle}{B'C'}=\frac{(C')^2+(B')^2-(A')^2}{2BC}=\frac{C^2+B^2-A^2}{2BC}=\alpha.$$

(SWS)⇒(SSW): Wlog. let $C' = C \geq B' = B$ and $\alpha' = \alpha$. From the law of cosines, one obtains $A' = A$ and subsequently $\gamma' = \gamma$.

(SSW)⇒(SWW): Wlog. let $C' = C \geq B' = B$ and $\gamma' = \gamma$. This time, the law of cosines yields a quadratic equation $A^2 + B^2 = C^2 + 2\gamma AB$ in A. Because $C \geq B$, only the solution

$$A=\gamma B+\underbrace{\sqrt{(\gamma^2-1)B^2+C^2}}_{\geq|\gamma|B}=B'+\sqrt{((\gamma')^2-1)(B')^2-(C')^2}=A'$$

is possible. With the law of cosines, one now obtains either $\alpha' = \alpha$ or $\beta' = \beta$.

(SWW)⇒(K): Let $\alpha' = \alpha$ and $\beta' = \beta$. According to Lemma 7.35, $\gamma' = \gamma$. So let wlog. $C' = C$. After a translation, we can assume $a' = a = (0, 0)$. After a rotation, $b' = b = C(1, 0)$ holds. Let $c = B(x, y)$. Then $\alpha = \langle (x, y), (1, 0)\rangle = x$ and $y = \pm\sqrt{1 - x^2}$ holds. Therefore, c lies either on the line $G := \mathbb{R}(\alpha, \sqrt{1 - \alpha^2})$ or on the line $\mathbb{R}(\alpha, -\sqrt{1 - \alpha^2})$. By reflecting across the x-axis if necessary, we may assume the first case. Now let $b - c = A(x, y)$. Then $\beta = \langle (1, 0), (x, y)\rangle = x$ and $y = \pm\sqrt{1 - \beta^2}$. Since c has a positive y-coordinate, $y = -\sqrt{1 - \beta^2}$ must hold. Thus c lies on the line $H := b + \mathbb{R}(\beta, -\sqrt{1 - \beta^2})$. Overall, $c = G \wedge H = c'$, i.e., Δ and Δ' are congruent. □

Theorem 7.37 (Similarity theorems)**.** *For triangles Δ and Δ', the following statements are equivalent:*

- *(S) Δ and Δ' are similar.*
- *(WW) Two angles of Δ and Δ' coincide.*
- *(SSS) The ratios of the side lengths of Δ and Δ' coincide.*
- *(SWS) One angle and the ratio of the adjacent sides coincide.*
- *(SSW) A side ratio and the angle opposite the larger side coincide.*

Proof. We use the notation as in Theorem 7.36.

(Ä)⇒(WW): Let $f = (v, S) \in \mathrm{Aut}(\mathcal{E})$ with $S^{\mathrm{t}}S = \lambda 1_2$ and $f(\Delta) = \Delta'$. Then $\langle a' - c', b' - c'\rangle = \langle S(a-c), S(b-c)\rangle = \lambda\langle a-c, b-c\rangle$ holds. From this, the claim follows.

(WW)⇒(SSS): From Lemma 7.4 it follows that

$$\frac{1-\alpha^2}{1-\beta^2} = \frac{[a-b, a-c]^2}{B^2C^2}\frac{A^2C^2}{[b-a, b-c]^2} = \frac{A^2}{B^2}.$$

This shows $A'/A = B'/B = C'/C$.

(SSS)⇒(SWS): Since angles are preserved by scaling, we can assume $A' = A$, $B' = B$ and $C' = C$. According to the (SSS) congruence theorem, Δ and Δ' are congruent and therefore also have identical angles.

(SWS)⇒(SSW): Let $A'/A = B'/B$ and $\gamma' = \gamma$. After scaling, we can again assume $A' = A$ and $B' = B$. Now Δ and Δ' are congruent according to the (SWS) congruence theorem.

(SSW)⇒(Ä): This time the claim follows from the (SSW) congruence theorem. ☐

Lemma 7.38. *A point $x \in \mathcal{P}$ lies on the circumcircle of a triangle a, b, c if and only if*

$$[a,b,c]|x|^2 = [x,b,c]|a|^2 + [a,x,c]|b|^2 + [a,b,x]|c|^2$$

holds.

Proof. Since the equation is satisfied for $x = a, b, c$, it suffices to show that the equation describes a circle. Since a, b, c are in general position, $[a,b,c] \neq 0$ holds. We can therefore divide by $[a,b,c]$. Using $[x,b,c] = \langle x, b^\perp\rangle + [b,c] + \langle c^\perp, x\rangle$ etc., one obtains the equivalent equation $|x|^2 = 2\langle x, m\rangle - |m|^2 + \rho^2$ for suitable $m \in \mathcal{P}$ and $\rho \in \mathbb{R}$. This describes exactly $K_\rho(m)$. ☐

Lemma 7.39. *For $a, b, c, d, x \in \mathcal{P}$,*

$$\kappa_{abcd} := [a,b,c]|x-d|^2 - [b,c,d]|x-a|^2 + [c,d,a]|x-b|^2 - [d,a,b]|x-c|^2$$

does not depend on x.

Proof. It holds that

$$\begin{aligned}[a,b,c] - [b,c,d] + [c,d,a] - [d,a,b] &= [a,b] + [b,c] + [c,a] - [b,c] - [c,d] - [d,b] + [c,d] \\ &\quad + [d,a] + [a,c] - [d,a] - [a,b] - [b,d] = 0.\end{aligned}$$

Because of $|x-d|^2 = |x| - 2\langle x, d\rangle$, the $|x|^2$-terms cancel out. Because of $[a,b]d + [b,d]a + [d,a]b = 0$ (Lemma 5.10), the linear terms in x also cancel out. ☐

Definition 7.40. A *quadrilateral* consists of four points $a, b, c, d \in \mathcal{P}$, no three of which are collinear.

Theorem 7.41 (Ptolemy)**.** *Let $a, b, c, d \in \mathcal{P}$ be a quadrilateral such that a, b (resp. b, c) lie on the same side of cd (resp. ad). Then the following are equivalent:*

(1) a, b, c, d lie on a circle.

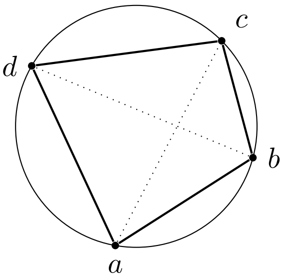
(2) The angles at a and b are equal (with respect to c, d).

(3) The angles at a and c add up to 0.

(4) $\kappa_{abcd} = 0$.

(5) $|a-b||c-d| + |a-d||b-c| = |a-c||b-d|$.

If applicable, a, b, c, d is called a cyclic quadrilateral.



Proof. (1)⇒(2): Since the triangles a, c, d and b, c, d have the same circumcircle (radius), it holds that

$$\frac{\langle a-c, a-d\rangle^2}{|a-c|^2|a-d|^2} = 1 - \frac{[a,c,d]^2}{|a-c|^2|a-d|^2} = 1 - \frac{[b,c,d]^2}{|b-c|^2|b-d|^2} = \frac{\langle b-c, b-d\rangle^2}{|b-c|^2|b-d|^2}.$$

Since the mapping $x \mapsto \frac{\langle x-c, x-d\rangle^2}{|x-c|^2|x-d|^2}$ is continuous on the circular arc between a and b, it must even be constant. This shows the claim.

(2)⇒(1): The above calculation shows that a, c, d and b, c, d have the same circumradius. Through the points c, d, the circumcenter is already uniquely determined. Therefore, a, b, c, d lie on the common circumcircle.

(1)⇒(3): As in (1)⇒(2), one shows that the angles at a and c are equal in absolute value. According to (2), we may shift c on the circle such that ac passes through the center of the circle. According to Thales, the angles at b and d are equal to 0. There now exist $\lambda, \mu \in \mathbb{R}$ with $\lambda(a-d)^\perp = c - d$, $\mu(a-b)^\perp = c - b$, $\lambda > 0$ and $\mu < 0$ (counter-clockwise rotation). This shows

$$\langle c-b, c-d\rangle = \lambda\mu\langle a-c, a-d\rangle.$$

Therefore, the angles at a and c have different signs and add up to 0.

(3)⇒(1): Analogous to (2)⇒(1).

(1)⇒(4): If one chooses x to be the circumcenter in Lemma 7.39, then $\kappa_{abcd} = 0$ follows.

(4)⇒(5): We show (independently of (4))

$$\begin{aligned}4\kappa_{abcd}^2 = {} & (|a-b||c-d| + |a-c||b-d| + |a-d||b-c|) \\ & (|a-b||c-d| + |a-c||b-d| - |a-d||b-c|) \\ & (|a-b||c-d| - |a-c||b-d| + |a-d||b-c|) \\ & (-|a-b||c-d| + |a-c||b-d| + |a-d||b-c|).\end{aligned} \tag{7.1}$$

After translation, we may assume $d = 0$. A calculation shows

$$|a-b||c| = |a||b||c|\Big|\frac{a}{|a|^2} - \frac{b}{|b|^2}\Big|.$$

The first product in 7.1 therefore has the form

$$|a||b||c|\Big(\big|\frac{a}{|a|^2} - \frac{b}{|b|^2}\big| + \big|\frac{b}{|b|^2} - \frac{c}{|c|^2}\big| + \big|\frac{c}{|c|^2} - \frac{a}{|a|^2}\big|\Big).$$

Using Heron's formula for the triangle $\frac{a}{|a|^2}, \frac{b}{|b|^2}, \frac{c}{|c|^2}$, the right side of (7.1) is equal to

$$4|a|^4|b|^4|c|^4\Big[\frac{a}{|a|^2}, \frac{b}{|b|^2}, \frac{c}{|c|^2}\Big] = 4\big|[a,b]|c|^2 + [b,c]|a|^2 + [c,a]|b|^2\big|^2 = 4\kappa_{abc0}^2.$$

According to (4), one of the four products is now 0. According to the triangle inequality for $\frac{a}{|a|^2}, \frac{b}{|b|^2}, \frac{c}{|c|^2}$, the points are collinear. Since a, c lie on different sides of bd, $\big|\frac{a}{|a|^2} - \frac{b}{|b|^2}\big| + \big|\frac{b}{|b|^2} - \frac{c}{|c|^2}\big| = \big|\frac{a}{|a|^2} - \frac{c}{|c|^2}\big|$ must hold. Thus, the third product in (7.1) vanishes.

(5)⇒(4): Follows from (7.1).

(4)⇒(1): With $x = 0$ in κ_{abcd}, the claim follows from Lemma 7.38. □

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