On the complement of a union of cosets

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Abstract

Let $g_1H_1, \ldots, g_nH_n$ be left cosets of subgroups of a group G whose union U is a proper subset of G. We prove that G is the union of at most 2^n left translates of $A := G \setminus U$. For finite G this yields $|A| \geq |G|/2^n$, which settles a conjecture of Tărnăuceanu and the author. Equality $|A| = |G|/2^n$ can only hold when A is a left coset of $K := H_1 \cap \ldots \cap H_n$. Moreover, every subgroup occurring in an irredundant cover of an arbitrary group by n cosets has index at most 2^{n-1}. This improves a lemma of Neumann.

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1 Introduction

A finite group G is never the union of two proper subgroups H_1 and H_2, and an easy count shows more precisely that $H_1 \cup H_2$ misses at least a quarter of the elements of G. Results of this kind have a long history. Cameron and Cohen [2] showed that at least $|H|$ elements lie outside a union of conjugates of a proper subgroup $H \leq G$, and the groups covered by a prescribed number of proper subgroups have been classified in many cases (see the survey [1]).

The general question behind these results is the following. Let $C_1, \ldots, C_n$ be cosets of subgroups of a finite group G, and suppose that

$$A := G \setminus (C_1 \cup \ldots \cup C_n)$$

is nonempty. How small can A be? In [7] Tărnăuceanu and the author proved the bound $|A| \geq |G|/(2n!)$ and conjectured that

$$|A| \geq \frac{|G|}{2^n} \tag{1.1}$$

(this is Problem 21.115 in the Kourovka notebook [3]). I learned from Stefanos Aivazidis that an AI-generated proof of (1.1) has been announced on GitHub [6]. The aim of this paper is to prove the following stronger result for arbitrary (possibly infinite) groups.

Theorem 1. *Let G be a group with (left or right) cosets $C_1, \ldots, C_n$ of subgroups such that $A := G \setminus (C_1 \cup \ldots \cup C_n) \neq \varnothing$. Then there exist $t_1, \ldots, t_m \in G$ with $G = t_1A \cup \ldots \cup t_mA$ and $m \leq 2^n$.*

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For finite G the union bound $|G| \leq m|A| \leq 2^n|A|$ yields (1.1). It has been pointed out in [7] that (1.1) is best possible for $G = \mathbb{F}_2^n$ and $C_i := \{x \in G : x_i = 1\}$. The constant 2^n in Theorem 1 is sharp for infinite groups as well: In $G = \mathbb{Z}^n$ let $C_i := \{x \in G : x_i \text{ odd}\}$. Then $A = (2\mathbb{Z})^n$ is a subgroup of index 2^n, so exactly 2^n translates of A are needed to cover G.

The next result describes the equality case for finite G in general. We may assume w.l.o.g. that the C_i are left cosets since $Hg = g(g^{-1}Hg)$ for $g \in G$ and $H \leq G$.

Theorem 2. *Let G be a finite group with left cosets $C_i = g_iH_i$ of subgroups $H_i \leq G$ such that $A = G \setminus (C_1 \cup \ldots \cup C_n) \neq \varnothing$. Let $K := H_1 \cap \ldots \cap H_n$. Then*

$$|A| = \frac{|G|}{2^n} \quad \Longleftrightarrow \quad A = aK \text{ for some } a \in G \text{ and } |G : K| = 2^n.$$

In this case the subgroups $H_1, \ldots, H_n$ are pairwise distinct.

Now we turn to the situation where $A = \varnothing$. Recall that a *cover* $G = C_1 \cup \ldots \cup C_n$ by cosets is called *irredundant* if no C_i can be omitted. A classical lemma of Neumann [5] asserts that all subgroups occurring in an irredundant cover have finite index, and that $|G : H_i| \leq n$ holds for at least one i. In [4] he bounded *all* the indices by $|G : H_i| \leq u_n$, where $u_1 = 1$ and $u_{k+1} = u_k^2 + u_k$, a bound of order $c^{2^{n-1}}$ with $c \approx 1.5979$. We improve this doubly exponential bound to the sharp exponential bound 2^{n-1}.

Theorem 3. *Let $G = C_1 \cup \ldots \cup C_n$ be an irredundant cover of a group G by left cosets $C_i = g_iH_i$. Then $|G : H_i| \leq 2^{n-1}$ for $i = 1, \ldots, n$. This bound is best possible for every n.*

We remark that Tomkinson [8] proved the sharp bound $|G : H_1 \cap \ldots \cap H_n| \leq n!$ in the situation of Theorem 3.

Throughout, $[n] := \{1, \ldots, n\}$.

2 A Helly lemma for unions of equivalence classes

Let $\mathcal{M}_n$ denote the space of multilinear polynomials in $\mathbb{Q}[X_1, \ldots, X_n]$, that is, the $\mathbb{Q}$-span of the 2^n monomials $\prod_{i \in S} X_i$ with $S \subseteq [n]$. These monomials are linearly independent, so

$$\dim_{\mathbb{Q}} \mathcal{M}_n = 2^n. \tag{2.1}$$

The following lemma establishes a Helly property for certain unions of equivalence classes.

Lemma 4. *Let X be a set equipped with equivalence relations $\sim_1, \ldots, \sim_n$ and let $U_1, \ldots, U_m \subseteq X$ be sets of the form*

$$U_j = Q_{j,1} \cup \ldots \cup Q_{j,n},$$

where $Q_{j,i}$ is a $\sim_i$-class. If $U_1 \cap \ldots \cap U_m = \varnothing$ while no proper subfamily has empty intersection, then $m \leq 2^n$. If moreover $\sim_i = \sim_j$ for some $i \neq j$, then $m < 2^n$.

Proof. By minimality there is, for every $j \in [m]$, an element

$$x_j \in \bigcap_{k \neq j} U_k,$$

and $x_j \notin U_j$ because the total intersection is empty. Only finitely many equivalence classes are relevant, namely the $Q_{j,i}$ and the classes of the x_j. Hence for each i we may choose an injection λ_i from the set of $\sim_i$-classes occurring here into $\mathbb{Q}$. Put

$$\alpha_{j,i} := \lambda_i(Q_{j,i}), \qquad \beta_{j,i} := \lambda_i([x_j]_{\sim_i}), \qquad \alpha_j := (\alpha_{j,1}, \ldots, \alpha_{j,n}) \in \mathbb{Q}^n.$$

Since $x_j \notin U_j$, the element x_j lies in none of the classes $Q_{j,1}, \ldots, Q_{j,n}$, whence

$$\beta_{j,i} \neq \alpha_{j,i} \qquad (1 \leq i \leq n). \tag{2.2}$$

On the other hand $x_j \in U_k$ for $k \neq j$, so x_j lies in $Q_{k,i}$ for at least one i and therefore

$$\beta_{j,i} = \alpha_{k,i} \quad \text{for some } i \in [n]. \tag{2.3}$$

Now define

$$p_j := \prod_{i=1}^{n} (X_i - \beta_{j,i}) \in \mathcal{M}_n \qquad (1 \leq j \leq m).$$

By (2.2) we have $p_j(\alpha_j) \neq 0$, and by (2.3) we have $p_j(\alpha_k) = 0$ for $k \neq j$. Evaluating a vanishing linear combination $\sum_j c_j p_j = 0$ at α_k isolates $c_k p_k(\alpha_k)$ and thus gives $c_k = 0$. Hence $p_1, \ldots, p_m$ are linearly independent in $\mathcal{M}_n$ and $m \leq 2^n$ by (2.1).

Suppose finally that $\sim_i = \sim_j$ with $i \neq j$. Then both relations have the same classes, so we may choose $\lambda_i = \lambda_j$ and obtain $\beta_{l,i} = \beta_{l,j}$ for every l. Consequently each p_l contains the factor $(X_i - \beta_{l,i})(X_j - \beta_{l,i})$ and is therefore symmetric in X_i and X_j. Now the p_l cannot span $\mathcal{M}_n$, so $m < 2^n$. □

3 Union of translates

Let G, H_i, $C_i = g_i H_i$ and A be as in Section 1. Set $U := C_1 \cup \ldots \cup C_n$. For each i let $\sim_i$ be the equivalence relation on G whose classes are the left H_i-cosets. Every left translate of U then has the shape required in Lemma 4, namely

$$tU = \bigcup_{i=1}^{n} t g_i H_i \qquad (t \in G). \tag{3.1}$$

Readers interested only in finite groups may jump directly to the proof of Theorem 1 below. For infinite G an additional ingredient is required.

For $\varnothing \neq S \subseteq [n]$ we write $H_S := \bigcap_{i \in S} H_i$. Set $\mathcal{L} := \{H_S : \varnothing \neq S \subseteq [n]\}$, a family of subgroups of G closed under intersection.

Lemma 5. *Let $D = dL$ and $D' = d'L'$ be left cosets with $L, L' \in \mathcal{L}$. Either $D \cap D' = \varnothing$ or $D \cap D'$ is a left coset of $L \cap L' \in \mathcal{L}$. In the latter case $D \cap D' = D$ if and only if $L \subseteq L'$.*

Proof. If $y \in D \cap D'$, then $D = yL$ and $D' = yL'$, hence $D \cap D' = y(L \cap L')$. This equals $D = yL$ if and only if $L \cap L' = L$. □

Lemma 6. *Suppose that $1 \notin U$. Then there is a finite subset $T \subseteq G$ with $\bigcap_{t\in T} tU = \varnothing$.*

Proof. For $L \in \mathcal{L}$ let the *height* $h(L)$ be the largest $k \geq 0$ such that there is a chain $L = L_0 \supsetneq L_1 \supsetneq \ldots \supsetneq L_k$ in $\mathcal{L}$. To a finite list $\mathcal{D} = (D_1, \ldots, D_r)$ of left cosets $D_\rho = d_\rho L_\rho$ with $L_\rho \in \mathcal{L}$ we attach the weight

$$w(\mathcal{D}) := \sum_{\rho=1}^{r} (n+1)^{h(L_\rho)} \in \mathbb{N}.$$

Note that $w(\mathcal{D}) = 0$ if and only if $\mathcal{D}$ is the empty list.

We construct finite sets T together with lists $\mathcal{D}$ whose union is $W := \bigcap_{t\in T} tU$, starting with $T = \{1\}$ and $\mathcal{D} = (g_1H_1, \ldots, g_nH_n)$. Assume T and $\mathcal{D}$ have been constructed and $W \neq \varnothing$, so that $\mathcal{D}$ is nonempty. Choose $x \in D_1$ and put $T' := T \cup \{x\}$ and $W' := W \cap xU$. Since $1 \notin U$ we have $x \notin xU$. We build a list $\mathcal{D}'$ for W' as follows. Every D_ρ with $D_\rho \subseteq xU$ is kept. Every other D_ρ is replaced by the nonempty sets among $D_\rho \cap xg_iH_i$ with $i \in [n]$, whose union is $D_\rho \cap xU$. Each of these is a proper subset of D_ρ, for otherwise $D_\rho \subseteq xg_iH_i \subseteq xU$. Hence by Lemma 5 it is a left coset of $L_\rho \cap H_i \in \mathcal{L}$ with $L_\rho \cap H_i \subsetneq L_\rho$, so its height is at most $h(L_\rho) - 1$. Consequently the contribution of ρ to the weight drops from $(n+1)^{h(L_\rho)}$ to at most $n(n+1)^{h(L_\rho)-1} < (n+1)^{h(L_\rho)}$. The coset D_1 is of the second kind, because $x \in D_1$ but $x \notin xU$. Therefore $w(\mathcal{D}') < w(\mathcal{D})$.

A strictly decreasing sequence of nonnegative integers is finite, so after finitely many steps the construction must halt, which happens only when $W = \varnothing$. □

Proof of Theorem 1. Fix $a \in A$ and replace A, U and every C_i by their left translates by a^{-1}. This changes neither the subgroups H_i nor the assertion, so we may assume $1 \in A$, that is, $1 \notin U$. By Lemma 6 there is a finite $T \subseteq G$ with $\bigcap_{t\in T} tU = \varnothing$ (for finite G, this is obvious since $t \notin tU$). Choose T inclusion-minimal with this property and write $T = \{t_1, \ldots, t_m\}$. By (3.1) the sets $t_1U, \ldots, t_mU$ satisfy the hypotheses of Lemma 4 with $X = G$, so $m \leq 2^n$. Taking complements and using $G \setminus t_jU = t_jA$ we obtain

$$G = t_1A \cup \ldots \cup t_mA.$$

□

4 The equality case

We now analyze when (1.1) is an equality and prove Theorem 2.

The proof of Theorem 2 uses the polynomials of Lemma 4 in the concrete situation at hand, so we make them explicit. Let G be finite and choose for every $i \in [n]$ an injection λ_i from the set of left H_i-cosets into $\mathbb{Q}$. For $y \in G$ and $t \in G$ put

$$p_y := \prod_{i=1}^{n} \bigl(X_i - \lambda_i(yH_i)\bigr) \in \mathcal{M}_n, \qquad \alpha_t := \bigl(\lambda_1(tg_1H_1), \ldots, \lambda_n(tg_nH_n)\bigr) \in \mathbb{Q}^n. \tag{4.1}$$

Since the λ_i are injective, $p_y(\alpha_t) = 0$ holds if and only if $yH_i = tg_iH_i$ for some i, that is, if and only if $y \in tU$. Hence

$$p_y(\alpha_t) = 0 \iff y \in tU, \qquad p_y(\alpha_t) \neq 0 \iff y \in tA. \tag{4.2}$$

Proof of Theorem 2. If $A = aK$ and $|G : K| = 2^n$, then $|A| = |K| = |G|/2^n$. Conversely, assume $|A| = |G|/2^n$. Fix $a \in A$ and replace all C_i by $a^{-1}C_i$. This changes neither $|A|$ nor the subgroups H_i, and it replaces A by $a^{-1}A \ni 1$. It therefore suffices to prove $A = K$ under the additional assumption $1 \in A$. Since $K \leq H_i$, we have $C_iK = g_iH_iK = g_iH_i = C_i$ for every i. This gives $UK = U$ and $AK = A$. As $1 \in A$, we conclude $K \subseteq A$.

Choose $T \subseteq G$ inclusion-minimal with $\bigcap_{t\in T} tU = \varnothing$, which is possible by Lemma 6. Equivalently, $G = \bigcup_{t\in T} tA$ and no proper subfamily covers G. By (3.1) and Lemma 4 we have $|T| \leq 2^n$, whereas

$$|G| \leq \sum_{t\in T} |tA| = |T|\,|A| = |T|\,\frac{|G|}{2^n}$$

forces $|T| \geq 2^n$. Hence $|T| = 2^n$ and the above estimate is an equality, which means that

$$G = \bigcup_{t\in T} tA \quad \text{is a disjoint union.} \tag{4.3}$$

For every $t \in T$ pick $y_t \in tA$. By (4.3) we have $y_t \notin sA$ for $s \in T \setminus \{t\}$, so (4.2) yields

$$p_{y_t}(\alpha_t) \neq 0, \qquad p_{y_t}(\alpha_s) = 0 \quad (s \in T \setminus \{t\}).$$

As in the proof of Lemma 4 the polynomials p_{y_t} with $t \in T$ are linearly independent, and since there are $2^n = \dim \mathcal{M}_n$ of them, they form a basis of $\mathcal{M}_n$.

Let $t_0 \in T$ be fixed, and let $y \in t_0A$ be arbitrary. Again by (4.3) and (4.2) we have $p_y(\alpha_s) = 0$ for $s \in T \setminus \{t_0\}$. Writing $p_y = \sum_{t\in T} c_t p_{y_t}$ and evaluating at α_s gives $c_s p_{y_s}(\alpha_s) = p_y(\alpha_s) = 0$, hence $c_s = 0$ for every $s \neq t_0$. Thus $p_y = c\, p_{y_{t_0}}$ for some $c \in \mathbb{Q}$. Comparing in (4.1) the coefficients of $X_1 \cdots X_n$ gives $c = 1$, and comparing the coefficients of $\prod_{i\neq j} X_i$ gives

$$\lambda_j(yH_j) = \lambda_j(y_{t_0}H_j) \qquad (1 \leq j \leq n).$$

Since λ_j is injective, this means $yH_j = y_{t_0}H_j$ for all j, that is, $y \in y_{t_0}K$. As $y \in t_0A$ was arbitrary, $t_0A \subseteq y_{t_0}K$ and therefore

$$|A| = |t_0A| \leq |K|.$$

Combined with $K \subseteq A$ this gives $A = K$, and $|G : K| = |G|/|A| = 2^n$.

If $H_i = H_j$ for some $i \neq j$, then $\sim_i = \sim_j$ and Lemma 4 gives $|T| < 2^n$, contradicting $|T| = 2^n$. Hence the H_i are pairwise distinct. □

5 Irredundant covers

Proof of Theorem 3. Fix $i \in [n]$ and put $A_i := G \setminus \bigcup_{j\neq i} C_j$. Irredundancy gives $A_i \neq \varnothing$ and the covering property gives $A_i \subseteq C_i$. Applying Theorem 1 to the $n-1$ cosets C_j with $j \neq i$ we find $t_1, \dots, t_m \in G$ with $m \leq 2^{n-1}$ and

$$G = \bigcup_{k=1}^{m} t_kA_i \subseteq \bigcup_{k=1}^{m} t_kg_iH_i.$$

Thus G is a union of at most 2^{n-1} left cosets of H_i and $|G : H_i| \leq 2^{n-1}$.

For the sharpness let $G := \mathbb{F}_2^{n-1}$, let $C_i := \{x \in G : x_i = 1\}$ for $i < n$ and let $C_n := \{0\}$. Every nonzero $x \in G$ has a coordinate equal to 1, so these n cosets cover G. The i-th standard basis vector lies in C_i only, and 0 lies in C_n only, so the cover is irredundant. Here $H_n = 1$ and $|G : H_n| = 2^{n-1}$. □

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