Universally extendible characters

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Abstract

Let θ be an ordinary character or a Brauer character of a finite group N. We show that θ extends to every overgroup $G = G_\theta \trianglerighteq N$ if and only if $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$ and the deflated character $\overline{\theta}$ of $\mathrm{Inn}(N)$ extends to $\mathrm{Aut}(N)_\theta$. This equivalence is compatible with fields of values. A similar investigation for extensions from subnormal subgroups leads to a new characteristic subgroup of N.

1 Introduction

All groups in this paper are finite. An important problem in representation theory is to decide when a character θ of a normal subgroup $N \trianglelefteq G$ extends to G, i. e. when there exists a character χ of G with $\chi_N = \theta$. Necessarily, θ must be G-invariant. For irreducible θ the most prominent sufficient criteria make assumptions on the embedding $N \subseteq G$ and hold for ordinary characters and for Brauer characters alike:

- G/N is cyclic (see [8, Theorem (8.12)]),
- N is a Hall subgroup of G, i. e. $\gcd(|N|, |G : N|) = 1$ (see [8, Theorem (8.13)]),
- θ extends to all preimages of Sylow subgroups of G/N (see [8, Theorem (8.29)]),
- $\gcd(o(\theta)\theta(1), |G : N|) = 1$, where $o(\theta)$ denotes the determinantal order of θ (see [8, Theorem (8.23)]).

The aim of this paper is to derive a criterion independent of G, i. e. we characterize when θ extends to *every* possible overgroup.

Let $\mathrm{Char}_p(N)$ be the set of Brauer characters of N with respect to an algebraically closed field of characteristic $p > 0$. The irreducible Brauer characters are denoted by $\mathrm{IBr}_p(N)$ as usual. For convenience, we denote the set of (irreducible) ordinary complex characters by $\mathrm{Char}_0(N) := \mathrm{Char}(N)$ and $\mathrm{IBr}_0(N) := \mathrm{Irr}(N)$ respectively. In this context, every element is regarded as p-regular when $p = 0$. The kernel $\mathrm{Ker}(\theta)$ of $\theta \in \mathrm{Char}_p(N)$ is the intersection of the kernels of the irreducible constituents of θ.

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Definition 1.1. Let $\theta \in \mathrm{Char}_p(N)$ with $p \geq 0$. If N is a normal subgroup of a group G, we denote the stabilizer of θ in G by G_θ. We call G an *overgroup* of θ if θ is G-invariant, i. e. $G_\theta = G$ (for irreducible θ, (G, N, θ) is then a (modular) character triple in the notation of [8]). Moreover, θ is called *universally extendible* if θ extends to every overgroup, i. e. for every such G there exists a $\chi \in \mathrm{Char}_p(G)$ with $\chi_N = \theta$. Finally, N itself is called *universally extendible* if every ordinary character of N is universally extendible.

Recall that the inner automorphism group $\mathrm{Inn}(N)$ of N is naturally isomorphic to $N/\mathrm{Z}(N)$. Every character of N is a class function and, as such, stabilized by $\mathrm{Inn}(N)$. If $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$, θ *deflates* to a character $\overline{\theta}$ of $N/\mathrm{Z}(N) \cong \mathrm{Inn}(N)$. Our main theorem incorporates fields of values.

Theorem 1.2. *Let $F \subseteq \mathbb{C}$ be a field and $\theta \in \mathrm{Char}_p(N)$ with $p \geq 0$. Then θ has an extension with values in F to every overgroup if and only if $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$ and $\overline{\theta}$ has an extension to $\mathrm{Aut}(N)_\theta$ with values in F.*

Corollary 1.3. *A character θ of N is universally extendible if and only if $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$ and $\overline{\theta}$ has an extension to $\mathrm{Aut}(N)_\theta$.*

Proof. Take $F = \mathbb{C}$ in Theorem 1.2. □

It is further possible to remove the G-invariance requirement on θ. In fact, θ extends to every $G \trianglerighteq N$ if and only if $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$ and $\overline{\theta}$ extends to $\mathrm{Aut}(N)$.

It may happen that θ is not universally extendible, but 2θ is (Example 3.5((iii))). Recall that $\mathrm{Char}_p(N)$ can be identified with $\mathrm{Char}_0(N)$ whenever p does not divide $|N|$ (see [8, Theorem (2.12)]). In this situation, θ can be universally extendible as a Brauer character, but not as an ordinary character when $p \mid |\mathrm{Aut}(N)|$ (Example 3.2).

For abelian groups N only the multiples of 1_N are universally extendible. On the opposite end we have:

Corollary 1.4. *A group N is universally extendible if and only if $\mathrm{Z}(N) = 1$ and every $\chi \in \mathrm{Irr}(\mathrm{Inn}(N))$ extends to its stabilizer in $\mathrm{Aut}(N)$.*

Corollary 1.4 implies that complete groups N, like symmetric groups of degree ≥ 7, are universally extendible (here $\mathrm{Z}(N) = 1$ and $\mathrm{Out}(N) = 1$). This is generalized in Proposition 3.3 below. In Section 4 we study when extensions are afforded by a representation over a given field. Subsequently, we investigate extensions from subnormal subgroups in Section 5. This leads to a new characteristic subgroup of N, which we compute in special cases in Section 6.

2 Proofs of the main theorems

In the following we fix $\theta \in \mathrm{Char}_p(N)$ with $p \geq 0$. Set $A := \mathrm{Aut}(N)_\theta$ and $\mathrm{Out}(N)_\theta := A/\mathrm{Inn}(N)$. Let $\kappa\colon N \to \mathrm{Inn}(N)$, $x \mapsto \kappa_x$ be the conjugation map with $\kappa_x(y) = xyx^{-1}$ for $x, y \in N$. For $k \geq 1$ let $\delta\colon N \to N^k$, $x \mapsto (x, \ldots, x)$, be the diagonal map. For a group G, let $\exp(G)$ denote the exponent of G, i. e. the smallest integer $e \geq 1$ such that $g^e = 1$ for all $g \in G$. The symmetric and alternating groups of degree d are denoted by S_d and A_d respectively.

The following two elementary observations will be used several times.

Lemma 2.1. *Let $H \leq S_k$ be a transitive permutation group where k is divisible by $\exp(N/N')$. Then $\delta(N) \leq (N \wr H)'$.*

Proof. For $x \in N$ and $1 \leq i \leq k$ let $e_i(x) \in N^k$ be the element with x in position i and 1 elsewhere, so that $\delta(x) = e_1(x) \cdots e_k(x)$. Set $W := N \wr H$. For every i there is some $h \in H$ with $he_1(x)h^{-1} = e_i(x)$ and therefore $e_1(x)^{-1}e_i(x) = e_1(x)^{-1}he_1(x)h^{-1} \in W'$. Since $e_1(x), \ldots, e_k(x)$ commute pairwise, multiplying over $i = 1, \ldots, k$ yields

$$e_1(x)^{-k}\delta(x) = \prod_{i=1}^{k} e_1(x)^{-1}e_i(x) \in W'.$$

Finally $x^k \in N'$ by the choice of k, so that $e_1(x)^k = e_1(x^k) \in e_1(N') \leq W'$ and $\delta(x) \in W'$. □

Lemma 2.2. *Let $W := N \wr A_k$ where $k \geq 5$. Let $\psi \in \mathrm{IBr}_p(W)$ with $\psi(1) < k$. Then $\psi_{N^k} = \psi(1)(\nu \times \ldots \times \nu)$ for some linear $\nu \in \mathrm{IBr}_p(N)$. In particular, $\psi(x, 1, \ldots, 1) = \psi(1)\nu(x)$ for every p-regular $x \in N$. If k is divisible by $\exp(N/N')$, then $\psi(\delta(x)) = \psi(1)$.*

Proof. Since ordinary characters are Brauer characters with respect to some $p \nmid |N|$, we may assume that $p > 0$. By [8, Theorem (8.21)] the irreducible Brauer characters of N^k are the products $\nu_1 \times \ldots \times \nu_k$ with $\nu_i \in \mathrm{IBr}_p(N)$. Let $\mu := \nu_1 \times \ldots \times \nu_k$ be a constituent of ψ_{N^k}. By Clifford's theorem [8, Corollary (8.7)], ψ_{N^k} is a multiple of the sum of the A_k-orbit of μ, and therefore

$$|A_k : (A_k)_\mu|\mu(1) \leq \psi(1) < k.$$

Every proper subgroup of A_k has index at least k, since A_k is simple and $|A_k| > |S_{k-1}|$. Consequently, μ is A_k-invariant. Since A_k acts transitively on the k factors, we have $\nu_1 = \ldots = \nu_k =: \nu$. From $\nu(1)^k = \mu(1) < k \leq 2^k$ we deduce that ν is linear, so that $\mu(1) = 1$ and $\psi_{N^k} = \psi(1)\mu$. For p-regular $x \in N$ this gives $\psi(x, 1, \ldots, 1) = \psi(1)\nu(x)$ and $\psi(\delta(x)) = \psi(1)\nu(x)^k$. Finally, ν is trivial on N', whence $\nu^k = 1_N$ if k is a multiple of $\exp(N/N')$. □

Proof of Theorem 1.2. Suppose first that $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$ and $\overline{\theta}$ extends to $\chi \in \mathrm{Char}_p(A)$ with values in F. Let Δ be a representation of A affording χ. Let G be an overgroup of θ, and let $\gamma\colon G \to A$ be the conjugation map. Then $\Delta \circ \gamma$ is a representation of G affording $\chi \circ \gamma \in \mathrm{Char}_p(G)$, which satisfies $\chi(\gamma(x)) = \chi(\kappa_x) = \overline{\theta}(\kappa_x) = \theta(x)$ for p-regular $x \in N$. Hence, θ extends to G with values in F.

Assume conversely that θ has an extension with values in F to every overgroup. Let $\theta = \sum_{i=1}^{r} m_i\theta_i$ with distinct $\theta_1, \ldots, \theta_r \in \mathrm{IBr}_p(N)$ and $m_i \geq 1$, and set $m := \max\{m_1, \ldots, m_r\}$. Choose an integer $k \geq \max(5, m+1)$ divisible by $\exp(N/N')$, and let $T := A_k$ act naturally on $\Omega := \{1, \ldots, k\}$. Let $W := N \wr T = N^k \rtimes T$ be the corresponding wreath product. It is easy to check that A acts on the left of W via

$${}^{\alpha}(x_1, \ldots, x_k, t) := (\alpha(x_1), \ldots, \alpha(x_k), t).$$

Set

$$G := W \rtimes A \cong N^k \rtimes (T \times A)$$

and identify N^k, A, T and W with their canonical copies inside G. Let $\mu\colon N \to G$, $x \mapsto \delta(x^{-1})\kappa_x$. For $x, y \in N$ we compute

$$\mu(x)\mu(y) = \delta(x^{-1})\kappa_x\delta(y^{-1})\kappa_y = \delta(x^{-1})\delta(\kappa_x(y)^{-1})\kappa_x\kappa_y = \delta((xy)^{-1})\kappa_{xy} = \mu(xy),$$

i. e. μ is a homomorphism, and clearly injective. Moreover, $\alpha(\mu(x)) = \mu(\alpha(x))$ for $\alpha \in A$. Hence, A acts on $\mu(N)$ in the same way as it acts on N. Since $\delta(N)$ is centralized by T and T commutes with A, the subgroup $\mu(N)$ is centralized by T. For $x_1, \dots, x_k, y \in N$ we compute

$$\mu(y)(x_1, \dots, x_k) = (y^{-1}\kappa_y(x_1), \dots, y^{-1}\kappa_y(x_k), \kappa_y) = (x_1, \dots, x_k)\mu(y).$$

Hence, $\mu(N)$ is centralized by W. In particular, $\mu(N) \trianglelefteq G$. Since μ commutes with the action of A, we may replace G by an isomorphic copy in which μ becomes the inclusion map of N. Then G is an overgroup of θ, and $x = \delta(x^{-1})\kappa_x$, i. e. $\kappa_x = \delta(x)x$, for every $x \in N$.

By hypothesis, θ extends to some $\chi \in \mathrm{Char}_p(G)$ with values in F. Since $[N, W] = 1$, the map $\pi\colon N \times W \to NW$, $(u, w) \mapsto uw$ is a surjective homomorphism, so that $\widetilde{\chi} := \chi_{NW} \circ \pi$ lies in $\mathrm{Char}_p(N \times W)$. Thus, we can write

$$\widetilde{\chi} = \sum_{i=1}^{r} \sum_{\psi \in \mathrm{IBr}_p(W)} a_{i\psi}(\theta_i \times \psi)$$

with integers $a_{i\psi} \geq 0$, where only the constituents θ_i of θ occur since $\widetilde{\chi}_{N\times 1} = \theta$ and $(\theta_i \times \psi)_{N\times 1} = \psi(1)\theta_i$. Comparing the restrictions to $N \times 1$ and to $1 \times W$ yields

$$m_i = \sum_{\psi} a_{i\psi}\psi(1), \qquad \chi_W = \sum_{i=1}^{r} \theta_i(1) \sum_{\psi} a_{i\psi}\psi.$$

Hence, every irreducible constituent ψ of χ_W satisfies $\psi(1) \leq m_i \leq m < k$ for some i. Lemma 2.2 gives $\psi(\delta(x)) = \psi(1)$ for p-regular $x \in N$. Summing over the irreducible constituents of χ_W yields $\chi(\delta(x)) = \chi(1)$, i. e. $\delta(x) \in \mathrm{Ker}(\chi)$.

For p-regular $z \in \mathrm{Z}(N)$ we have $\kappa_z = 1$ and therefore $z = \delta(z^{-1}) \in \mathrm{Ker}(\chi)$. This shows $\theta(z) = \chi(z) = \chi(1) = \theta(1)$, i. e. $z \in \mathrm{Ker}(\theta)$. Since $\mathrm{Z}(N)$ is abelian, its p-elements (when $p > 0$) form a normal p-subgroup of N and therefore lie in $\mathrm{O}_p(N) \leq \mathrm{Ker}(\theta)$ (see [8, Lemma (2.32)]). Hence, $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$. More generally, $\delta(x) \in \mathrm{Ker}(\chi)$ implies $\chi(\kappa_x) = \chi(\delta(x)x) = \chi(x) = \theta(x)$ for p-regular $x \in N$. Thus, χ_A extends $\overline{\theta}$ on $\mathrm{Inn}(N)$ and takes only values of χ. □

Lemma 2.3. *Let G be an overgroup of $\theta \in \mathrm{Char}(N)$. If every irreducible constituent of θ extends to its stabilizer in G, then θ extends to G.*

Proof. Since θ is G-invariant, its irreducible constituents form G-orbits. Let $\theta_1, \dots, \theta_k \in \mathrm{Irr}(N)$ be representatives of those orbits. Let $m_i := [\theta, \theta_i]$ be the multiplicity of θ_i in θ. By hypothesis, each θ_i extends to some $\hat{\theta}_i \in \mathrm{Irr}(G_{\theta_i})$. Now $\chi := \sum_{i=1}^{k} m_i\hat{\theta}_i^G$ is an extension of θ because

$$\chi_N = \sum_{i=1}^{k} m_i(\hat{\theta}_i^G)_N = \sum_{i=1}^{k} m_i \sum_{G_{\theta_i}x \in G_{\theta_i}\backslash G} \theta_i^x = \sum_{\psi \in \mathrm{Irr}(N)} [\theta, \psi]\psi = \theta.$$ □

Proof of Corollary 1.4. If $\mathrm{Z}(N) \neq 1$, choose $\lambda \in \mathrm{Irr}(\mathrm{Z}(N))$ non-trivial and $\theta \in \mathrm{Irr}(N)$ lying over λ. Then $\mathrm{Z}(N) \nsubseteq \mathrm{Ker}(\theta)$ and θ is not universally extendible by Corollary 1.3. Now let $\mathrm{Z}(N) = 1$, so that $N = \mathrm{Inn}(N)$ and $\overline{\theta} = \theta$. By Corollary 1.3, an irreducible character of N is universally extendible if and only if it extends to its stabilizer in $\mathrm{Aut}(N)$. Hence, the claim follows from Lemma 2.3. □

We record one more property of the overgroup G built above.

Lemma 2.4. *There exists an overgroup G satisfying $N \cap G' = \kappa^{-1}(A')$.*

Proof. Let G be the overgroup constructed in the proof of Theorem 1.2 (observe that the construction does not require universal extendibility). Recall from the proof that $x = \delta(x^{-1})\kappa_x$ for $x \in N$. Since $W = N^k \rtimes T$ with T transitive on Ω and k is divisible by $\exp(N/N')$, Lemma 2.1 gives $\delta(x) \in W' \leq G'$ for every $x \in N$. Hence, $x \in G'$ if and only if $\kappa_x \in G'$. Since $G/W \cong A$, we have $\kappa_x \in G' \leq WA'$ if and only if $\kappa_x \in A'$. □

3 Consequences

Proposition 3.1. *Let p be a prime not dividing $|N|$ and let $\theta \in \mathrm{Irr}(N)$. Then θ is universally extendible as an ordinary character if and only if θ is universally extendible as a p-Brauer character.*

Proof. By Theorem 1.2 we may assume that $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$, since otherwise θ is universally extendible in neither sense. If θ has an ordinary extension to every overgroup, then the restrictions of these extensions to the p-regular classes are p-Brauer extensions by [8, Corollary (2.9)].

Conversely, let $\overline{\theta}$ extend to some $\chi \in \mathrm{Char}_p(A)$. Let q be a prime and let P be the preimage in A of a Sylow q-subgroup of $\mathrm{Out}(N)_\theta$. If $q \neq p$, then $|P| = |\mathrm{Inn}(N)||P : \mathrm{Inn}(N)|$ is not divisible by p. Hence, $\mathrm{Char}_p(P) = \mathrm{Char}(P)$, and χ_P is an ordinary extension of $\overline{\theta}$. If $q = p$, then p divides neither $\overline{\theta}(1)$ nor the determinantal order $o(\overline{\theta})$, because both divide $|\mathrm{Inn}(N)|$. Since $|P : \mathrm{Inn}(N)|$ is a power of p, the character $\overline{\theta}$ extends to P by [5, Corollary 8.16]. Now $\overline{\theta}$ extends to A by [5, Corollary 11.31] and the claim follows from Corollary 1.3. □

Example 3.2. Proposition 3.1 fails for reducible characters: there exists a group N of order $2 \cdot 3^{10}$ with a unique non-trivial linear character θ such that 2θ is universally extendible as a 5-Brauer character, but not as an ordinary character (see the GAP [2] code attached to this pdf).

We present some useful sufficient criteria for universal extendibility.

Proposition 3.3.

(a) Let $\theta \in \mathrm{IBr}_p(N)$ with $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$. If the Schur multiplier $H^2(\mathrm{Out}(N)_\theta, \mathbb{C}^\times)$ is a p-group (trivial if $p = 0$), then θ is universally extendible.

(b) If $\mathrm{Z}(N) = 1$ and every Sylow subgroup of $\mathrm{Out}(N)$ is cyclic, then N is universally extendible.

Proof.

(a) Let $B := \mathrm{Out}(N)_\theta$. Suppose first that $p > 0$. Let E be an algebraically closed field of characteristic p. By [8, Theorem (8.15)] the triple $(A, \mathrm{Inn}(N), \overline{\theta})$ has an obstruction class in $H^2(B, E^\times)$ which is trivial if and only if $\overline{\theta}$ extends to A. By [6, Proposition 2.1.14], $H^2(B, E^\times)$ is isomorphic to the p'-part of the Schur multiplier $H^2(B, \mathbb{C}^\times)$ and therefore trivial. Hence, $\overline{\theta}$ extends to A. If $p = 0$ and $H^2(B, \mathbb{C}^\times) = 1$, then $\overline{\theta}$ extends to A by [5, Theorem 11.7]. In both cases Theorem 1.2 applies.

(b) Let $\theta \in \mathrm{Irr}(N)$. Every Sylow subgroup of $\mathrm{Out}(N)_\theta$ is contained in a Sylow subgroup of $\mathrm{Out}(N)$ and is therefore cyclic. This implies that $H^2(\mathrm{Out}(N)_\theta, \mathbb{C}^\times) = 1$ by [5, Corollary 11.21], so θ is universally extendible by (a). Now the claim follows from Corollary 1.4 (cf. [5, Corollaries 11.22 and 11.31]). □

By [8, Problem (2.7)], the linear Brauer characters of N can be identified with $\mathrm{Irr}\big(N/N'\mathrm{O}^{p'}(N)\big)$, i. e. with the ordinary linear characters of p'-order (when $p > 0$).

Proposition 3.4. *For every linear character $\theta \in \mathrm{IBr}_p(N)$ the following are equivalent:*

(1) θ is universally extendible.

(2) $N \cap G' \leq \mathrm{Ker}(\theta)$ for every overgroup G of θ.

(3) $\kappa^{-1}(A') \leq \mathrm{Ker}(\theta)$.

Proof.

(1)⇒(2): An extension $\hat{\theta}$ of θ to an overgroup G has degree 1. Hence, $G' \leq \mathrm{Ker}(\hat{\theta})$ and $N \cap G' \leq \mathrm{Ker}(\theta)$.

(2)⇒(3): Apply (2) to the overgroup G of Lemma 2.4.

(3)⇒(1): By hypothesis, $\mathrm{Z}(N) = \mathrm{Ker}(\kappa) \leq \kappa^{-1}(A') \leq \mathrm{Ker}(\theta)$ and $\mathrm{Inn}(N) \cap A' \leq \mathrm{Ker}(\overline{\theta})$. By deflation, we may regard $\overline{\theta}$ as a linear character of $\mathrm{Inn}(N)/(\mathrm{Inn}(N) \cap A') \cong \mathrm{Inn}(N)A'/A'$. As such, it extends to the abelian group A/A'. By inflation, $\overline{\theta}$ extends to A. Hence, (1) follows from Corollary 1.3. □

We conclude with some examples.

Example 3.5.

(i) Let
$$N = \langle a, b \mid a^{12} = 1,\ b^2 = a^6,\ a^b = a^{-1} \rangle = \texttt{SmallGroup}(24, 4)$$
be the dicyclic group of order 24. Then $\mathrm{Z}(N) = \langle a^6 \rangle$ has order 2, and there is a unique $\theta \in \mathrm{Irr}(N)$ of degree 2 with $\mathrm{Ker}(\theta) = \mathrm{Z}(N)$, so that $\overline{\theta} \in \mathrm{Irr}(\mathrm{Inn}(N))$ is faithful. Since $\mathrm{Z}(N)$ is characteristic in N, we have $A = \mathrm{Aut}(N) \cong D_8 \times S_3$, but $\overline{\theta}$ does not extend to A. Consequently, θ is not universally extendible. Let θ^0 be the restriction of θ to the 2-regular elements. Since $\mathrm{IBr}_2(N) = \{1_N, \theta^0\}$, we have $\mathrm{Out}(N)_{\theta^0} = \mathrm{Out}(N) \cong C_2^2$, whose Schur multiplier C_2 has trivial $2'$-part. Hence, θ^0 is universally extendible by Proposition 3.3(a).

(ii) Let θ be the unique non-trivial linear character of $N \cong D_{10}$. Since $A = \mathrm{Aut}(N) \cong C_5 \rtimes C_4$, we have $\kappa^{-1}(A') = N' \leq \mathrm{Ker}(\theta)$, so θ is universally extendible. Note that θ is rational, but neither of the two extensions to A is rational (their field of values is $\mathbb{Q}(i)$).

(iii) Let θ be the unique non-trivial linear character of the generalized dihedral group $N = C_3^2 \rtimes C_2$. Here
$$A = \mathrm{Aut}(N) \cong \mathrm{AGL}(2,3) \cong C_3^2 \rtimes \mathrm{GL}(2,3)$$
is the affine linear group and $A' = \mathrm{ASL}(2,3)$. The inverting automorphism is the central involution of $\mathrm{SL}(2,3)$, so $\mathrm{Inn}(N) \leq A'$ and therefore $\kappa^{-1}(A') = N \not\leq \mathrm{Ker}(\theta)$. By Proposition 3.4, θ is not universally extendible. On the other hand, the inflation to A of any faithful $\chi \in \mathrm{Irr}(\mathrm{GL}(2,3))$ of degree 2 is an extension of 2θ. Hence, 2θ is universally extendible by Corollary 1.3. Finally, let $\lambda \in \mathrm{Irr}(N)$ be of degree 2. Then $\mathrm{Out}(N)_{\theta+\lambda} = \mathrm{Out}(N)_\lambda \cong S_3$ has trivial Schur multiplier. By [5, Theorem 11.7], $\overline{\theta}$ and $\overline{\lambda}$ both extend to $\mathrm{Aut}(N)_{\theta+\lambda}$. Therefore, $\theta + \lambda$ is universally extendible by Corollary 1.3.

4 Realizable characters over given fields

In this section we investigate when an extension $\chi \in \mathrm{Char}(G)$ of an ordinary character θ of N is *realizable* over a field $F \subseteq \mathbb{C}$, i. e. there exists an F-representation of G affording χ.

Theorem 4.1. *Let $\theta \in \mathrm{Char}(N)$ and let $F \subseteq \mathbb{C}$ be a field. Then θ has an extension realizable over F to every overgroup if and only if $\mathrm{Z}(N) \leq \mathrm{Ker}(\theta)$ and $\overline{\theta}$ has an extension to $\mathrm{Aut}(N)_\theta$ realizable over F.*

Proof. Both implications in the proof of Theorem 1.2 produce the desired representations. □

Recall that an irreducible character ψ is realizable over F if and only if $F(\psi) = F$ and the Schur index $m_F(\psi)$ equals 1 (see [5, Definition 10.1 and Corollary 10.2(d)]).

Lemma 4.2. *Let $H \leq G$ and let $\psi \in \mathrm{Irr}(H)$ be realizable over a field $F \subseteq \mathbb{C}$. Then an extension χ of ψ to G is realizable over F if and only if $F(\chi) = F$.*

Proof. One direction is trivial. So let $F(\chi) = F$. Since $\chi_H = \psi$ is irreducible, so is χ. Moreover, $F(\psi) \subseteq F(\chi) = F$ and $[\psi^G, \chi] = [\psi, \chi_H] = 1$ by Frobenius reciprocity. Now [5, Lemma 10.4] gives $m_F(\chi) \mid m_F(\psi) = 1$. Hence χ is realizable over F. □

By Lemma 4.2, the conditions in Theorem 1.2 and Theorem 4.1 are equivalent for every $\theta \in \mathrm{Irr}(N)$ realizable over F.

Example 4.3.

(i) Let $N \cong C_{15} \rtimes C_4 = \texttt{SmallGroup}(60, 7)$ with $\mathrm{Z}(N) = 1$ and $N' \cong C_{15}$. Let $\theta \in \mathrm{Irr}(N)$ be the unique character of degree 2 with $\mathrm{Ker}(\theta) \cong C_5$, that is, the inflation of the faithful character of the dicyclic group N/C_5 of order 12. Then θ has rational values and Frobenius–Schur indicator -1. Thus, $m_{\mathbb{Q}}(\theta) = 2$ and θ is not realizable over $\mathbb{Q}$. One can show that θ has an extension to $\mathrm{Aut}(N)$ with values in $\mathbb{Q}(i)$ and Schur index 1. Hence, θ has an extension realizable over $\mathbb{Q}(i)$ to every overgroup by Theorem 4.1.

(ii) Let $V := \mathbb{F}_{81}$, regarded as a vector space of dimension 4 over $\mathbb{F}_3$. Let $a \in \mathrm{GL}(V)$ be the multiplication by a generator of $\mathbb{F}_{81}^\times$. Let $b \in \mathrm{GL}(V)$ be the Frobenius map $x \mapsto x^3$. We set

$$H_0 := \langle a^4, a^2b^2 \rangle, \qquad H := \langle a^2, b^2 \rangle, \qquad N := V \rtimes H, \qquad K := V \rtimes H_0.$$

Then $H_0 \cong C_5 \rtimes C_8$, $H \cong D_{10} \times C_8$ and $|N| = 6480$. Moreover, $\mathrm{Z}(N) = 1$. Let $\theta \in \mathrm{Irr}(N)$ be the unique linear character with $\mathrm{Ker}(\theta) = K$. It takes the values ± 1 and is realizable over $\mathbb{Q}$. Here $A = \mathrm{Aut}(N)_\theta$ has order 25920 with $A/K \cong Q_8$ and $\mathrm{Z}(A/K) = N/K$. In particular, $2\overline{\theta}$ has a unique (rational) extension $\chi \in \mathrm{Char}(A)$. By Theorem 1.2, 2θ has rational extensions to every overgroup. However, by Theorem 4.1, there exists an overgroup to which 2θ has no extension realizable over $\mathbb{Q}$, since χ is not realizable over $\mathbb{Q}$. Therefore, the remark after Lemma 4.2 fails for reducible characters.

5 Subnormal subgroups

Throughout this section $\theta \in \mathrm{Char}_p(N)$ with $p \geq 0$. Recall that N is *subnormal* in a group G if there exists a chain $N = N_1 \trianglelefteq \ldots \trianglelefteq N_k = G$. In this case we write $N \trianglelefteq\trianglelefteq G$. Since normal subgroups are subnormal, one may ask which characters extend to every group G containing N as a subnormal subgroup. For non-normal N the natural analog of G-invariance is invariance under *G-fusion*, i. e. $\theta(x^g) = \theta(x)$ whenever $x, x^g \in N$ are p-regular and $g \in G$. We call θ *subnormally extendible* if θ extends to every group $G \trianglerighteq\trianglerighteq N$ in which θ is fusion invariant. If $N \trianglelefteq G$, then G-fusion invariance is just G-invariance. Clearly, every subnormally extendible character is universally extendible. We will see in Theorem 5.3 that the converse is false.

Theorem 5.1. *Let θ be subnormally extendible and let $N \trianglelefteq\trianglelefteq G$ such that θ is G-fusion invariant. Then θ is the restriction of a sum of linear characters of G. In particular, every irreducible constituent of θ is linear and extends to G. Moreover, $N \cap G' \leq \mathrm{Ker}(\theta)$.*

Proof. Choose an integer $k \geq \max(5, \theta(1) + 1)$ and let $H := G \wr A_k = G^k \rtimes A_k$, where A_k permutes the k factors naturally. We identify G with $G \times 1 \times \ldots \times 1$ and N with $N \times 1 \times \ldots \times 1$. A subnormal chain from N to G yields $N \trianglelefteq\trianglelefteq G \trianglelefteq G^k \trianglelefteq H$, so N is subnormal in H.

Let $x \in N$ and $h = (g_1, \ldots, g_k)t \in H$ with $g_1, \ldots, g_k \in G$ and $t \in A_k$. Conjugation by t moves the first component of G^k to position $t(1)$, and conjugation by $(g_1, \ldots, g_k)$ acts componentwise. Hence $hxh^{-1} \in N$ if and only if $t(1) = 1$ and $g_1 x g_1^{-1} \in N$, and in this case $hxh^{-1} = g_1 x g_1^{-1}$. Thus H-fusion of elements of N is G-fusion, and θ is H-fusion invariant. By hypothesis there exists $\chi \in \mathrm{Char}_p(H)$ with $\chi_N = \theta$.

Write $\chi = \sum_\psi a_\psi \psi$ with $\psi \in \mathrm{IBr}_p(H)$ and integers $a_\psi \geq 0$. Every ψ with $a_\psi > 0$ satisfies $\psi(1) \leq \chi(1) = \theta(1) < k$, so Lemma 2.2, applied to G in place of N, provides a linear $\nu_\psi \in \mathrm{IBr}_p(G)$ with $\psi(g) = \psi(1)\nu_\psi(g)$ for every p-regular $g \in G$. Hence

$$\chi_G = \sum_\psi a_\psi \psi(1) \nu_\psi$$

is a sum of linear characters in $\mathrm{IBr}_p(G)$, and $\theta = \chi_N = \sum_\psi a_\psi \psi(1)(\nu_\psi)_N$ is a sum of the linear characters $(\nu_\psi)_N \in \mathrm{IBr}_p(N)$. Since $\mathrm{IBr}_p(N)$ is linearly independent, every irreducible constituent of θ has the form $(\nu_\psi)_N$ and therefore extends to the linear character ν_ψ of G. As $G' \leq \mathrm{Ker}(\nu_\psi)$, we obtain $N \cap G' \leq \mathrm{Ker}(\theta)$. □

A linear character $\theta \in \mathrm{IBr}_p(N)$ is G-fusion invariant if and only if the *focal subgroup*

$$\mathrm{foc}_G(N) := \langle x^{-1} x^g : x, x^g \in N,\, g \in G \rangle$$

lies in $\mathrm{Ker}(\theta)$. Note that $N' \leq \mathrm{foc}_G(N) \leq N \cap G'$.

We will show next that fusion invariance is dispensable.

Lemma 5.2. *Let $N \trianglelefteq\trianglelefteq G$. Then there exists a group $H \trianglerighteq\trianglerighteq N$ such that $\mathrm{foc}_H(N) = N'$ and $N \cap G' \leq N \cap H'$.*

Proof. Let $U := N/N'$, $k := \exp(U)$ and $\pi\colon N \to U$ be the natural map. Let $V := U^k$ and $W := V \rtimes \langle t \rangle \cong U \wr C_k$. Set $H := W \times G$ and

$$f\colon N \to H, \qquad x \mapsto \big(\delta(\pi(x)), x\big).$$

Then f is an injective homomorphism, and $\pi(yxy^{-1}) = \pi(x)$ gives

$$(v,y)f(x)(v,y)^{-1} = \big(\delta(\pi(x)), yxy^{-1}\big) = f(yxy^{-1})$$

for $x, y \in N$ and $v \in V$. Hence $f(N) \trianglelefteq V \times N$, and a subnormal chain from N to G yields $V \times N \trianglelefteq\trianglelefteq V \times G \trianglelefteq H$, so that $f(N)$ is subnormal in H. We identify N with $f(N)$.

Since V is abelian and t fixes $\delta(U)$ pointwise, $\delta(U)$ lies in the center of W. For $x \in N$ and $h = (w,g) \in H$ we therefore have $hf(x)h^{-1} = \big(\delta(\pi(x)), gxg^{-1}\big)$, and this element lies in $f(N)$ only if $gxg^{-1} \in N$ and $\pi(gxg^{-1}) = \pi(x)$, i. e. $x^{-1}gxg^{-1} \in N'$. Thus $\mathrm{foc}_H(N) = N'$. As $\langle t \rangle$ permutes the coordinates of V transitively, Lemma 2.1 yields $\delta(U) \times 1 \le W' \times 1 \le H'$, and $1 \times G' \le H'$. Hence every $x \in N \cap G'$ satisfies $f(x) = \big(\delta(\pi(x)), 1\big)(1,x) \in H'$. □

Subnormal extendibility turns out to be governed by a single characteristic subgroup of N. We write

$$R(N) := \big\langle N \cap G' : N \trianglelefteq\trianglelefteq G \big\rangle \le N.$$

Let $N \trianglelefteq\trianglelefteq G$ and $\omega\colon N \to G$ be the inclusion map. For $\alpha \in \mathrm{Aut}(N)$, $\omega \circ \alpha^{-1}$ is another inclusion of N as a subnormal subgroup of G. Since

$$(\omega\alpha^{-1})^{-1}(N \cap G') = \alpha(\omega^{-1}(N \cap G')) = \alpha(N \cap G'),$$

$R(N)$ is invariant under α. Hence $R(N)$ is characteristic in N. Taking $G = N$ gives $N' \le R(N)$, so that $N/R(N)$ is abelian.

The following main theorem is the analog of Proposition 3.4.

Theorem 5.3. *For $\theta \in \mathrm{Char}_p(N)$ the following assertions are equivalent:*

(1) θ is subnormally extendible.

(2) θ extends to every group $G \trianglerighteq\trianglerighteq N$.

(3) Every irreducible constituent of θ is subnormally extendible.

(4) $R(N) \le \mathrm{Ker}(\theta)$.

Proof.

(1)⇒(4): Let θ be subnormally extendible and let $N \trianglelefteq\trianglelefteq G$. Since θ is invariant under N-fusion, Theorem 5.1 with $G = N$ gives $N' \le \mathrm{Ker}(\theta)$. Let H be as in Lemma 5.2. Since $\mathrm{foc}_H(N) = N' \le \mathrm{Ker}(\theta)$, the character θ is H-fusion invariant, and Theorem 5.1 yields $N \cap G' \le N \cap H' \le \mathrm{Ker}(\theta)$. As G was arbitrary, $R(N) \le \mathrm{Ker}(\theta)$.

(4)⇒(2): Let $R(N) \le \mathrm{Ker}(\theta)$ and let $N \trianglelefteq\trianglelefteq G$. Since $N' \le R(N) \le \mathrm{Ker}(\theta)$, every irreducible constituent λ of θ is linear, and $N \cap G' \le R(N) \le \mathrm{Ker}(\lambda)$. The deflation of λ to $N/(N \cap G') \cong NG'/G'$ extends to the abelian group G/G'. Hence, λ extends to a linear character $\hat{\lambda} \in \mathrm{IBr}_p(G)$. If $\theta = \sum_\lambda m_\lambda \lambda$, then $\sum_\lambda m_\lambda \hat{\lambda}$ is a sum of linear characters of G extending θ.

(2)⇒(1): This is clear.

(3)⇔(4): Since $\mathrm{Ker}(\theta)$ is the intersection of the kernels of the irreducible constituents of θ, condition (4) holds for θ if and only if it holds for every irreducible constituent of θ. Now apply the equivalence (1)⇔(4) to the constituents. □

In contrast to Example 3.2, the subnormal extendibility does not depend on the prime by Theorem 5.3.

It is natural to ask when θ extends to every group G containing N as an arbitrary subgroup in which θ is fusion invariant.

Proposition 5.4. *Suppose that θ extends to every group G which contains N as a subgroup and in which θ is G-fusion invariant. Then $\theta = \theta(1)1_N$.*

Proof. Choose an integer $k \geq \max(5, \theta(1)+1)$ divisible by $\exp(N/N')$ and let $W := N \wr T = N^k \rtimes T$, where $T := A_k$ permutes the k factors naturally. We identify N with $\delta(N) \leq W$.

Let $x, y \in N$ and $w = (y_1, \ldots, y_k)t \in W$ with $y_1, \ldots, y_k \in N$ and $t \in T$ such that $\delta(x)^w = \delta(y)$. The components of $\delta(x)^w$ are $x^{y_1}, \ldots, x^{y_k}$ in some order, so y is conjugate to x in N and $\theta(y) = \theta(x)$. Hence, every character in $\mathrm{Char}_p(N)$ is W-fusion invariant, and by hypothesis there is some $\chi \in \mathrm{Char}_p(W)$ with $\chi_N = \theta$.

Every irreducible constituent ψ of χ satisfies $\psi(1) \leq \chi(1) = \theta(1) < k$, so that $\psi(\delta(x)) = \psi(1)$ for p-regular $x \in N$ by Lemma 2.2. Summing over the irreducible constituents of χ yields $\theta(x) = \chi(\delta(x)) = \chi(1) = \theta(1)$ for every p-regular $x \in N$. □

In the next section we will show that non-trivial subnormally extendible characters actually exist.

6 Computing $R(N)$

For a given N, it is not obvious how to compute $R(N)$. In this section we derive a lower bound and compute $R(N)$ in special situations. Let $1 = \mathrm{Z}_0(N) \leq \mathrm{Z}_1(N) \leq \ldots$ be the upper central series of N and let $\mathrm{Z}_\infty(N)$ be its last term, the hypercenter of N.

Lemma 6.1. *For $k \geq 1$ and $i \geq 0$ we have $\delta(N)\mathrm{Z}_i(N)^k \trianglelefteq \delta(N)\mathrm{Z}_{i+1}(N)^k$. In particular, $\delta(N)$ is subnormal in $\delta(N)\mathrm{Z}_\infty(N)^k$.*

Proof. Let $\overline{N} := N/\mathrm{Z}_i(N)$ and let $\pi\colon N^k \to \overline{N}^k$ be induced by the natural map $N \to \overline{N}$. Since $\mathrm{Z}_{i+1}(N)/\mathrm{Z}_i(N) = \mathrm{Z}(\overline{N})$, we have $\delta(N)\mathrm{Z}_i(N)^k = \pi^{-1}(\delta(\overline{N}))$ and $\delta(N)\mathrm{Z}_{i+1}(N)^k = \pi^{-1}(\delta(\overline{N})\mathrm{Z}(\overline{N})^k)$. As $\mathrm{Z}(\overline{N})^k$ centralizes $\delta(\overline{N})$, the claim follows by taking preimages. □

Theorem 6.2. *We have $\mathrm{Z}_\infty(N)\kappa^{-1}(\mathrm{Aut}(N)') \leq R(N)$.*

Proof. Set $Z := \mathrm{Z}_\infty(N)$, $k := \exp(Z/Z')$ and $W := N \wr C_k = N^k \rtimes \langle t \rangle$, where $t(x_1, \ldots, x_k)t^{-1} = (x_k, x_1, \ldots, x_{k-1})$. Then $L := \delta(N)Z^k$ is normalized by t. Define

$$G := L\langle t \rangle \leq W.$$

By Lemma 6.1, $\delta(N)$ is subnormal in $L \trianglelefteq G$. Moreover $Z^k \rtimes \langle t \rangle \leq G$ and $\langle t \rangle$ permutes the k coordinates of Z^k transitively, so Lemma 2.1 yields $\delta(Z) \leq (Z^k \rtimes \langle t \rangle)' \leq G'$. Identifying N with $\delta(N)$ we obtain $Z \leq N \cap G' \leq R(N)$.

Applied to the trivial character of N, where $A = \mathrm{Aut}(N)$, Lemma 2.4 provides a group $G \trianglerighteq N$ with $N \cap G' = \kappa^{-1}(\mathrm{Aut}(N)')$, so that also $\kappa^{-1}(\mathrm{Aut}(N)') \leq R(N)$. □

Corollary 6.3. *If N is nilpotent, then $R(N) = N$. In particular, only the multiples of the trivial character are subnormally extendible.*

Proof. Since $\mathrm{Z}_\infty(N) = N$, Theorem 6.2 gives $R(N) = N$. Hence, the claim follows from Theorem 5.3. □

Example 6.4.

(i) Let $V := \mathbb{F}_5^3$, $d := \mathrm{diag}(2, 2, 4) \in \mathrm{GL}(3, 5)$ and

$$N := V \rtimes \langle d \rangle = \texttt{SmallGroup}(500, 47).$$

It can be shown that

$$\mathrm{Z}_\infty(N)\kappa^{-1}(\mathrm{Aut}(N)') = V\langle d^2 \rangle < N.$$

Let $D \cong C_4^3$ be the group of diagonal matrices in $\mathrm{GL}(3, 5)$, and let σ be the permutation matrix of a 3-cycle. Then N is subnormal in

$$G := V \rtimes (D \rtimes \langle \sigma \rangle),$$

and another computation shows that $N = N \cap G' = R(N)$. Hence, Theorem 6.2 can be strict.

(ii) Although $\mathrm{Z}_\infty(N)$ is nilpotent, the Fitting subgroup $\mathrm{F}(N)$ is not always contained in $R(N)$. For $N = \texttt{SmallGroup}(768, 1083540)$ one can show that $\mathrm{F}(N) = \mathrm{O}_2(N)$ has index 3 while $R(N) = \mathrm{O}^2(N)$ has index 2. We omit the details.

Now we prepare to construct a family of examples where $R(N) < N$. We first record how a subnormal subgroup interacts with coprime action on $\mathrm{O}_p(G)$.

Lemma 6.5. *Let U be a subnormal subgroup of a group G, let p be a prime and let $S \leq U$ be a p'-subgroup. Then $[\mathrm{O}_p(G), S] = [U \cap \mathrm{O}_p(G), S]$.*

Proof. Set $P := \mathrm{O}_p(G)$, $J_0 := UP$ and $J_{i+1} := U^{J_i}$ for $i \geq 0$. Since U is subnormal in J_0, we have $J_n = U$ for some $n \geq 0$. Moreover $U \leq J_{i+1} \leq J_i$, so $Q_i := J_i \cap P$ satisfies $Q_0 = P$ and $Q_n = U \cap P$. From $[Q_i, S] \leq [J_i, U] \leq U^{J_i} = J_{i+1}$ and $[Q_i, S] \leq P$ we obtain $[Q_i, S] \leq Q_{i+1}$. The coprime action of S on Q_i yields

$$[Q_i, S] = [[Q_i, S], S] \leq [Q_{i+1}, S] \leq [Q_i, S]$$

by [7, 8.2.7]. Hence $[Q_i, S] = [Q_{i+1}, S]$ for all i, and $[P, S] = [Q_0, S] = [Q_n, S] = [U \cap P, S]$. □

Theorem 6.6. *Suppose that N' is a cyclic Hall subgroup of N. Then $R(N) = \mathrm{C}_N(N')$.*

Proof. Set $D := N'$ and choose a complement S of D in N, which exists by the Schur–Zassenhaus theorem. Since D is abelian and $S \cong N/D$ is abelian, the group $N/[D, S]$ is generated by the central subgroup $D/[D, S]$ and the abelian subgroup $S[D, S]/[D, S]$, hence abelian. This gives $D = N' \leq [D, S]$, and $D = \mathrm{C}_D(S) \times [D, S]$ by coprime action [7, 8.4.2], so that $\mathrm{C}_D(S) = 1$. Dedekind's lemma yields $\mathrm{C}_N(D) = D\mathrm{C}_S(D)$, and if $z = xy \in \mathrm{Z}(N)$ with $x \in D$ and $y \in \mathrm{C}_S(D)$, then $x \in \mathrm{C}_D(S) = 1$. Therefore

$$\mathrm{Z}(N) = \mathrm{C}_S(D), \qquad \mathrm{C}_N(D) = D \times \mathrm{Z}(N). \tag{6.1}$$

Restriction to the characteristic subgroup D is a homomorphism from $\mathrm{Aut}(N)$ into the abelian group $\mathrm{Aut}(D)$, so $\mathrm{Aut}(N)'$ centralizes D and $\mathrm{Aut}(N)' \cap \mathrm{Inn}(N) \leq \kappa(\mathrm{C}_N(D))$. By (6.1) we have $\mathrm{C}_N(D) = D \times \mathrm{Z}(N) = \kappa^{-1}(\kappa(D))$, and $\kappa(D) = \mathrm{Inn}(N)' \leq \mathrm{Aut}(N)'$. Hence $\kappa^{-1}(\mathrm{Aut}(N)') = \mathrm{C}_N(D)$, and Theorem 6.2 gives $\mathrm{C}_N(D) \leq R(N)$.

For the reverse inclusion let G be a group containing N as a subnormal subgroup. Since $D = N' \leq G'$, Dedekind's lemma gives $N \cap G' = D(S \cap G')$. By (6.1) it therefore suffices to show that $S \cap G' \leq \mathrm{C}_S(D_p)$ for every prime divisor p of $|D|$, where $D_p \in \mathrm{Syl}_p(D)$.

As D_p is characteristic in D and D is characteristic in N, the group D_p is a subnormal p-subgroup of G and therefore $D_p \leq P := \mathrm{O}_p(G)$. Since $|N : D|$ is coprime to p, we have $D_p \in \mathrm{Syl}_p(N)$, and $N \cap P$ is a normal p-subgroup of N, whence $N \cap P = D_p$. Moreover $D_p = \mathrm{C}_{D_p}(S) \times [D_p, S]$ by [7, 8.4.2] and $\mathrm{C}_{D_p}(S) \leq \mathrm{C}_D(S) = 1$, so $[D_p, S] = D_p$. Thus Lemma 6.5 gives

$$[P, S] = [D_p, S] = D_p.$$

Set $V := P/\Phi(P)$. As $\Phi(P)$ is characteristic in $P \trianglelefteq G$, conjugation induces a representation $\rho\colon G \to \mathrm{GL}(V)$ over $\mathbb{F}_p$. Here, $[V, S]$ is the image $\overline{D_p}$ of $[P, S] = D_p$ in V. Since S acts coprimely on P and $[P, S] \neq 1$, [7, 8.2.9] shows that $\overline{D_p} \neq 1$. As $\overline{D_p}$ is a non-trivial quotient of the cyclic group D_p, we have $\overline{D_p} \cong D_p/\Phi(D_p)$ as $\mathbb{F}_pS$-modules. Now $V = \mathrm{C}_V(S) \oplus [V, S]$ by [7, 8.4.2], so $\det \rho(y)$ is the scalar by which $y \in S$ acts on $D_p/\Phi(D_p)$. The kernel of the natural map $\mathrm{Aut}(D_p) \to \mathrm{Aut}(D_p/\Phi(D_p))$ is a p-group by [7, 8.2.9] and $p \nmid |S|$, so $\det \rho(y) = 1$ if and only if $y \in \mathrm{C}_S(D_p)$. Since $\mathbb{F}_p^\times$ is abelian, $G' \leq \mathrm{Ker}(\det \circ \rho)$ and therefore $S \cap G' \leq \mathrm{C}_S(D_p)$. □

Example 6.7. If all Sylow subgroups of N are cyclic, then N' is cyclic of order coprime to $|N : N'|$ by a theorem of Zassenhaus [4, Satz IV.2.11], so Theorem 6.6 applies to every such N. For $N \cong D_{2n}$ with n odd, we obtain $R(N) = \mathrm{C}_N(N') \cong C_n$. Hence, the non-trivial linear character of N is subnormally extendible. In particular, $R(S_3) = A_3$. This implies an observation by P. Hall: S_3 is not subnormal in a perfect group (see [1, Theorem 3]).

The invariant $R(N)$ can also be estimated by passing to quotients. Recall that a *Fitting class* is a non-empty class $\mathfrak{F}$ of finite groups, closed under isomorphisms, which contains with every group G all normal subgroups of G, and with any two normal subgroups $N_1, N_2 \trianglelefteq G$ in $\mathfrak{F}$ also their product N_1N_2. Every finite group G then has a largest normal $\mathfrak{F}$-subgroup, the $\mathfrak{F}$-*radical* $G_\mathfrak{F}$. For instance, the π-groups for a set of primes π, the nilpotent groups and the solvable groups form Fitting classes. The corresponding radicals are $\mathrm{O}_\pi(G)$, the Fitting subgroup $\mathrm{F}(G)$ and the solvable radical of G. Since $N_\mathfrak{F}$ is characteristic in N, an easy induction on the length of a subnormal chain shows that $N_\mathfrak{F} \leq G_\mathfrak{F}$ whenever $N \trianglelefteq\trianglelefteq G$.

Proposition 6.8. *For every Fitting class $\mathfrak{F}$, we have $R(N)N_\mathfrak{F}/N_\mathfrak{F} \leq R(N/N_\mathfrak{F})$.*

Proof. Let $N \trianglelefteq\trianglelefteq G$ and set $P := G_\mathfrak{F}$. By the remark above, $N_\mathfrak{F} = N \cap P$. Hence $xN_\mathfrak{F} \mapsto xP$ is an isomorphism from $N/N_\mathfrak{F}$ onto NP/P, and NP/P is subnormal in G/P, being the image of N under the natural map.

Since R is compatible with isomorphisms, this isomorphism maps $R(N/N_\mathfrak{F})$ onto $R(NP/P)$. For $x \in N \cap G'$ we have $xP \in NP/P \cap (G/P)' \leq R(NP/P)$, so that $xN_\mathfrak{F} \in R(N/N_\mathfrak{F})$. Hence $(N \cap G')N_\mathfrak{F}/N_\mathfrak{F} \leq R(N/N_\mathfrak{F})$. Since G was arbitrary, the claim follows. □

Example 6.9.

(i) For $N = S_4$ we have $\mathrm{O}_2(N) = V \cong C_2^2$, $N/V \cong S_3$ and $R(S_3) = A_3$ by Example 6.7. Since the preimage of A_3 in S_4 is A_4, Proposition 6.8 gives $R(S_4) \leq A_4 = S_4' \leq R(S_4)$. Hence $R(S_4) = A_4$.

(ii) Let N be an indecomposable solvable complete group. By [3, Corollary B and p. 436], every group $G \trianglerighteq\trianglerighteq N$ satisfies $N \cap G' \leq N'\mathrm{F}(N)$. Therefore, $R(N) \leq N'\mathrm{F}(N)$. This generalizes (i).

In the examples so far N was solvable. For almost simple groups N the invariant $R(N)$ can be computed inside $\mathrm{Aut}(S)$, where S is the socle of N.

Theorem 6.10. *Let N be almost simple with socle S and regard N as a subgroup of $\mathrm{Aut}(S)$ via conjugation. Then*

$$R(N) = \big\langle N \cap B' : N \trianglelefteq\trianglelefteq B \leq \mathrm{Aut}(S) \big\rangle.$$

In particular, $R(N) = S$ if $\mathrm{Out}(S)$ is abelian, and $R(N) = N \cap \mathrm{Aut}(S)'$ if $N \trianglelefteq\trianglelefteq \mathrm{Aut}(S)$.

Proof. The right-hand side is contained in $R(N)$ by definition. Conversely, let $N \trianglelefteq\trianglelefteq G$. We construct a group B with $N \trianglelefteq\trianglelefteq B \leq \mathrm{Aut}(S)$ and $N \cap G' \leq N \cap B'$.

Since S is a subnormal non-abelian simple subgroup of G, every conjugate S^g with $g \in G$ is a component of G. By [7, 6.5.2], $S^g \leq N$ or $[S^g, N] = 1$. In the first case S^g and S are components of N, so $S^g = S$ or $[S^g, S] = 1$ by [7, 6.5.3], and the latter would give $S^g \leq \mathrm{C}_N(S) = 1$. Hence, N normalizes every conjugate of S and centralizes every conjugate different from S.

Let $G_1 := \mathrm{N}_G(S)$ and let $\rho\colon G_1 \to \mathrm{Aut}(S)$ be the conjugation map, so that ρ restricts to the given embedding of $N \leq G_1$. Set $B := \rho(G_1)$. As N is subnormal in G_1, its image N is subnormal in $B \leq \mathrm{Aut}(S)$. Let $1 = g_1, g_2, \ldots, g_r$ be a right transversal of G_1 in G, so that $S^{g_1}, \ldots, S^{g_r}$ are the distinct conjugates of S. Let $x \in N$. Since x normalizes S^{g_i}, we have $g_i x g_i^{-1} \in G_1$, i. e. x fixes every coset $G_1 g_i$ under right multiplication. Hence the transfer homomorphism $\tau\colon G \to G_1/G_1'$ satisfies

$$\tau(x) = \prod_{i=1}^{r} g_i x g_i^{-1} G_1'$$

(see [7, 7.1.1] and formula (5) there). For $i \geq 2$ the element x centralizes S^{g_i}, so $g_i x g_i^{-1}$ centralizes S and $\rho(g_i x g_i^{-1}) = 1$. Since $\rho(G_1') = B'$, the map ρ induces a homomorphism $\overline{\rho}\colon G_1/G_1' \to B/B'$ with $\overline{\rho}(\tau(x)) = \rho(x)B'$. If $x \in N \cap G'$, then $\tau(x) = 1$, because G_1/G_1' is abelian, and consequently $\rho(x) \in B'$. This shows $N \cap G' \leq N \cap B'$.

If $\mathrm{Out}(S)$ is abelian, then every B with $N \trianglelefteq\trianglelefteq B \leq \mathrm{Aut}(S)$ satisfies $S = S' \leq B' \leq S$, since B/S is abelian. Hence $R(N) = S$ in this case. If $N \trianglelefteq\trianglelefteq \mathrm{Aut}(S)$, then $B = \mathrm{Aut}(S)$ is admissible, and $N \cap B' \leq N \cap \mathrm{Aut}(S)'$ for every admissible B. □

Corollary 6.11. *For $n \geq 3$ we have $R(S_n) = A_n$. In particular, the sign character is subnormally extendible.*

Proof. For $n \in \{3, 4\}$ this is Example 6.7 and Example 6.9(i). For $n \geq 5$, S_n is almost simple with socle A_n. Since $\mathrm{Out}(A_n)$ is abelian (of order at most 4), $R(S_n) = A_n$ by Theorem 6.10. The second claim follows from Theorem 5.3. □

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References

[1] R. S. Dark, *On subnormal embedding theorems for groups*, J. London Math. Soc. **43** (1968), 387–390.

[2] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.16.1*; 2026, `https://www.gap-system.org`.

[3] H. Heineken and J. C. Lennox, *The subnormal embedding of complete groups*, J. Algebra **90** (1984), 435–445.

[4] B. Huppert, *Endliche Gruppen. I*, Grundlehren der Mathematischen Wissenschaften, Vol. 134, Springer-Verlag, Berlin, 1967.

[5] I. M. Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006.

[6] G. Karpilovsky, *The Schur multiplier*, London Mathematical Society Monographs. New Series, Vol. 2, The Clarendon Press Oxford University Press, New York, 1987.

[7] H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Universitext, Springer-Verlag, New York, 2004.

[8] G. Navarro, *Characters and blocks of finite groups*, London Mathematical Society Lecture Note Series, Vol. 250, Cambridge University Press, Cambridge, 1998.