GROUPS SATISFYING GASCHÜTZ'S COMPLEMENT THEOREM

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Abstract. A classical theorem of Gaschütz states that an abelian normal subgroup N of a finite group G has a complement in G whenever it has a complement in some subgroup H with $N \le H \le G$ and $\gcd(|N|, [G:H]) = 1$. We characterize the finite groups N for which this theorem remains valid without the abelian hypothesis: this is the case if and only if $\mathrm{Z}(N) \cap N' = 1$ and N has a complement in a quotient of the holomorph $\mathrm{Hol}(N)$ by a diagonal copy of N'. This completes a previous paper by the second author.

1. Introduction

All groups considered in this paper are finite. A subgroup C of a group G is a *complement* of a normal subgroup $N \trianglelefteq G$ if $G = NC$ and $N \cap C = 1$. Deciding whether a normal subgroup has a complement (that is, whether G splits over N) is a classical problem of group theory. For abelian N, a well-known theorem of Gaschütz reduces this question to a subgroup of coprime index.

Theorem (Gaschütz [2])**.** *Let N be an abelian normal subgroup of a group G, and let $N \le H \le G$ with $\gcd(|N|, [G:H]) = 1$. Then N has a complement in G if and only if N has a complement in H.*

This result was extended to groups N with abelian Sylow subgroups by the second author in [3, Theorem 2].[1] Following this paper, we say that *Gaschütz's theorem holds for* N, and write $\mathcal{G}(N)$, if the theorem holds without conditions on N: whenever $N \trianglelefteq G$, $N \le H \le G$, $\gcd(|N|, [G:H]) = 1$, and N has a complement in H, then N has a complement in G. By [3, Theorem 12], $\mathcal{G}(N)$ implies $\mathrm{Z}(N) \cap N' = 1$, where N' denotes the commutator subgroup of N. In particular, $\mathcal{G}(N)$ fails for all non-abelian nilpotent groups. For metabelian groups N, $\mathcal{G}(N)$ is in fact equivalent to $\mathrm{Z}(N) \cap N' = 1$ ([3, Corollary 14]).

As indicated in [3, Introduction], Gaschütz and Brandis both expressed their wish to find a complete intrinsic characterization of $\mathcal{G}(N)$. This is the goal of the present paper. For $x \in N$, let γ_x be the inner automorphism of N induced by x, i.e., $\gamma_x(y) := xyx^{-1}$ for $y \in N$. Let $\mathrm{Hol}(N) := N \rtimes \mathrm{Aut}(N)$ be the holomorph of N.

Theorem A. *Let N be a group and*

$$D := \left\{(d^{-1}, \gamma_d) : d \in N'\right\} \trianglelefteq \mathrm{Hol}(N).$$

Then Gaschütz's theorem holds for N if and only if $\mathrm{Z}(N) \cap N' = 1$ and N has a complement in $\mathrm{Hol}(N)/D$, where N is identified with its natural image in this quotient.

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[1] Note that the arXiv version of this paper has a different numbering for the theorems.

For abelian N, the two conditions hold trivially, in accordance with Gaschütz's theorem. For perfect groups N, we recover [3, Theorem 15]: $\mathcal{G}(N)$ holds if and only if $\mathrm{Z}(N) = 1$ and $\mathrm{Inn}(N)$ has a complement in $\mathrm{Aut}(N)$.

The proof consists of two reductions of independent interest. In Section 2 we show that $\mathcal{G}(N)$ is equivalent to the condition that every extension B of N which splits modulo the commutator subgroup N' already splits over N (Theorem 2.1). The proof turns any extension violating this condition into a Gaschütz counterexample $N \le H \trianglelefteq G$ with $[G : H] = p$, for any prime $p \nmid |B|$. In Section 3 we prove that, under the necessary condition $\mathrm{Z}(N) \cap N' = 1$, every extension of N which splits modulo N' is a fiber product of the single extension $\mathrm{Hol}(N)/D$ of $\mathrm{Aut}(N)/\gamma(N')$ (Theorem 3.1).

In the last section, we give an explicit example by answering [3, Problem 18]: $\mathcal{G}(N)$ does not hold for $N \cong (C_3^2 \rtimes Q_8) \times C_2$.

2. Splitting modulo the commutator subgroup

The property $\mathcal{G}(N)$ quantifies over pairs of groups $N \le H \le G$. Our first reduction replaces these pairs by single extensions of N and shows that only the splitting behavior modulo the commutator subgroup matters. This naturally extends Gaschütz's original theorem.

Theorem 2.1. *For every group N, the following conditions are equivalent.*

(i) *$\mathcal{G}(N)$ holds.*

(ii) *For every group B with $N \trianglelefteq B$, if N/N' has a complement in B/N', then N has a complement in B.*

Moreover, if condition (ii) *fails for B, then every prime $p \nmid |B|$ gives groups $N \le H \trianglelefteq G$ such that $G/H \cong C_p$ and N has a complement in H but not in G.*

Proof. Suppose first that condition (ii) holds, and let $N \le H \le G$ satisfy the hypotheses defining $\mathcal{G}(N)$, say with a complement K of N in H. Since N' is characteristic in $N \trianglelefteq G$, we have $N' \trianglelefteq G$, and N/N' is an abelian normal subgroup of G/N'. Dedekind's modular law gives $KN' \cap N = (K \cap N)N' = N'$, and $(KN')N = KN = H$; hence KN'/N' is a complement of N/N' in H/N'. Since $|N/N'|$ divides $|N|$ and $[G/N' : H/N'] = [G : H]$, these two quantities are coprime, so Gaschütz's theorem gives a complement of N/N' in G/N'. Applying condition (ii) with $B = G$ shows that N has a complement in G.

Suppose conversely that condition (ii) fails. We construct a witness G for the failure of $\mathcal{G}(N)$, similar to [3, proof of Theorem 12]. Choose $N \trianglelefteq B$ such that N/N' has a complement in B/N' but N has no complement in B, and put $R \coloneqq B/N$. Let $\pi\colon B \to R$ be the quotient map, choose a prime $p \nmid |B|$, and let

$$P \coloneqq \{(b_1, \dots, b_p) \in B^p : \pi(b_1) = \cdots = \pi(b_p)\}$$

be the p-fold fiber product of π with itself. The cyclic shift σ of P, given by $(b_1, \dots, b_p) \mapsto (b_p, b_1, \dots, b_{p-1})$, is an automorphism of order p; put $Q \coloneqq P \rtimes \langle\sigma\rangle$. The homomorphism $P \to R$ sending each tuple to its common value $\pi(b_1) = \cdots = \pi(b_p)$ is invariant under σ, so it extends to a homomorphism $\tau\colon Q \to R$ with $\tau(\sigma) = 1$. Form the fiber products

$$G \coloneqq B \times_R Q = \{(b, q) \in B \times Q : \pi(b) = \tau(q)\}, \qquad H \coloneqq B \times_R P = G \cap (B \times P),$$

and let $\rho\colon G \to Q$, $(b, q) \mapsto q$, be the projection onto the second coordinate. Given $q \in Q$, the surjectivity of π provides some $b \in B$ with $\pi(b) = \tau(q)$; hence ρ is surjective. We identify N with $\ker(\rho) = N \times 1 \trianglelefteq G$. Since $H = \rho^{-1}(P)$ and $Q/P \cong C_p$, we obtain $H \trianglelefteq G$ and $G/H \cong C_p$. Note that $\gcd(|N|, p) = 1$ because $|N|$ divides $|B|$.

Step 1: N has a complement in H.

Let $\lambda\colon P \to B$ be the projection onto the first coordinate and put

$$K \coloneqq \{(\lambda(x), x) : x \in P\} \leq B \times P.$$

Every $x \in P$ satisfies $\pi(\lambda(x)) = \tau(x)$, so $K \leq H$. If $(\lambda(x), x) \in N \times 1$, then $x = 1$ and hence $\lambda(x) = 1$; thus $K \cap N = 1$. Moreover, if $(b, x) \in H$, then $\pi(b) = \tau(x) = \pi(\lambda(x))$, so $n \coloneqq b\lambda(x)^{-1}$ lies in $\ker \pi = N$ and $(b, x) = (n, 1)(\lambda(x), x) \in NK$. Hence $H = NK$, and K is a complement of N in H.

Aiming at a contradiction, we assume from now on that N has a complement M in G.

Step 2: there is a homomorphism $\theta\colon Q \to B$ with $\pi\theta = \tau$.

By assumption, $M \cap \ker \rho = M \cap (N \times 1) = 1$ and $\rho(M) = \rho\big(M(N \times 1)\big) = \rho(G) = Q$, so ρ restricts to an isomorphism $M \to Q$. Composing its inverse $Q \to M$ with the projection $G \to B$ onto the first coordinate yields a homomorphism $\theta\colon Q \to B$ such that

$$M = \{(\theta(q), q) : q \in Q\} \leq B \times Q.$$

Since $M \leq G$, the defining equation of the fiber product forces $\pi(\theta(q)) = \tau(q)$ for all $q \in Q$.

Step 3: the coordinate copies of N have a common image under θ.

For $1 \leq i \leq p$, let $\varepsilon_i\colon N \to P$ place $n \in N$ in the ith coordinate and 1 in all others; the image lies in P because $\pi(n) = 1$. Inside Q we have $\sigma\varepsilon_i(n)\sigma^{-1} = \varepsilon_{i+1}(n)$, with indices taken modulo p. The order of $\theta(\sigma)$ divides the order p of σ as well as $|B|$. Thus, $\theta(\sigma) = 1$ as $\gcd(p, |B|) = 1$. This yields $\theta\varepsilon_{i+1} = \theta\varepsilon_i$ (composition of maps). Hence $\psi \coloneqq \theta\varepsilon_i\colon N \to B$ does not depend on i.

Step 4: $\psi(N') = 1$.

Tuples supported in distinct coordinates commute, so for all $m, n \in N$ (using $p \geq 2$),

$$\psi(m)\psi(n) = \theta(\varepsilon_1(m))\theta(\varepsilon_2(n)) = \theta\big(\varepsilon_1(m)\varepsilon_2(n)\big) = \theta\big(\varepsilon_2(n)\varepsilon_1(m)\big) = \psi(n)\psi(m).$$

Thus $\psi(N)$ is abelian and $N' \leq \ker \psi$.

Step 5: conclusion.

Let $\delta\colon B \to P$, $b \mapsto (b, \dots, b)$, be the diagonal embedding. For $u \in N'$ we have $\delta(u) = \varepsilon_1(u) \cdots \varepsilon_p(u)$ and therefore $\theta(\delta(u)) = \psi(u)^p = 1$ by Step 4. Hence the homomorphism $\theta\delta\colon B \to B$ is trivial on N'. Moreover,

$$\pi(\theta(\delta(b))) = \tau(\delta(b)) = \pi(b) \tag{1}$$

for all $b \in B$; that is, $\theta\delta$ leaves the image in R unchanged.

By the choice of B, the subgroup N/N' has a complement in B/N'; let $L \leq B$ be its inverse image, so that $N' \leq L$, $B = NL$ and $N \cap L = N'$. The last two equalities give $|B| = |N|\,|L| \,/\, |N'|$ and hence $[L : N'] = |B| \,/\, |N| = |R|$. Put $C \coloneqq \theta\delta(L)$. Since $N' \leq \ker(\theta\delta)$, we obtain $|C| \leq [L : N'] = |R|$. On the other hand,

$$\pi(C) = \pi(L) = \pi(LN) = \pi(B) = R$$

by (1). Consequently, π restricts to an isomorphism $C \to R$. In particular $C \cap N = C \cap \ker \pi = 1$, and $|NC| = |N|\,|C| = |N|\,|R| = |B|$ yields $B = NC$. Thus C is a complement of N in B, contradicting the choice of B. Therefore N has no complement in G. $\square$

We remark that Theorem 2.1 does not provide a new proof of Gaschütz's theorem because we made use of the original theorem in the proof.

3. Proof of the main theorem

Let N be a group and put $A \coloneqq \operatorname{Aut}(N)$. The inner automorphism map $\gamma\colon N \to A$, $x \mapsto \gamma_x$, is a homomorphism with image $\operatorname{Inn}(N)$ and kernel $\mathrm{Z}(N)$. Since N' is characteristic in N, the subgroup $\gamma(N')$ is normal in A. We use the left-action convention for $\operatorname{Hol}(N) = N \rtimes A$, so that

$$(n, \alpha)(m, \beta) = (n\alpha(m), \alpha\beta) \qquad (n, m \in N,\ \alpha, \beta \in A).$$

The map $N \to N \rtimes A$, $x \mapsto (x^{-1}, \gamma_x)$ is a monomorphism, since for $x, y \in N$

$$(x^{-1}, \gamma_x)(y^{-1}, \gamma_y) = \bigl(x^{-1}\gamma_x(y^{-1}), \gamma_x\gamma_y\bigr) = \bigl(y^{-1}x^{-1}, \gamma_{xy}\bigr) = \bigl((xy)^{-1}, \gamma_{xy}\bigr).$$

In particular,

$$D \coloneqq \bigl\{(d^{-1}, \gamma_d) : d \in N'\bigr\} \le \operatorname{Hol}(N)$$

is a subgroup isomorphic to N'. It centralizes $N \times 1$, because for $n \in N$ the products $(n, 1)(d^{-1}, \gamma_d)$ and $(d^{-1}, \gamma_d)(n, 1)$ are both equal to (nd^{-1}, γ_d). Conjugation by $(1, \alpha)$ with $\alpha \in A$ sends (d^{-1}, γ_d) to $(\alpha(d)^{-1}, \gamma_{\alpha(d)})$ with $\alpha(d) \in N'$. Since $N \rtimes A$ is generated by $N \times 1$ and $1 \times A$, we conclude that $D \trianglelefteq \operatorname{Hol}(N)$.

Theorem 3.1. *Let N be a group satisfying $\mathrm{Z}(N) \cap N' = 1$. Define*

$$U_N \coloneqq \operatorname{Hol}(N)/D, \qquad T_N \coloneqq A/\gamma(N').$$

The following holds:

(i) *N embeds as a normal subgroup of U_N.*

(ii) *There is an epimorphism $\pi\colon U_N \to T_N$ with kernel N.*

(iii) *N/N' has a complement in U_N/N'.*

(iv) *If $N \trianglelefteq B$ and N/N' has a complement in B/N', then B is a fiber product of π and some homomorphism $B/N \to T_N$. Here, N is identified with $N \times 1$.*

Proof.

(i) An element (d^{-1}, γ_d) of D belongs to $N \times 1$ if and only if $\gamma_d = 1$, that is, if and only if $d \in \mathrm{Z}(N) \cap N' = 1$. Hence $(N \times 1) \cap D = 1$. We can therefore identify N with $(N \times 1)D/D \trianglelefteq U_N$.

(ii) Since $(n, 1)(d^{-1}, \gamma_d) = (nd^{-1}, \gamma_d)$ for $n \in N$ and $d \in N'$, we have $(N \times 1)D = N \rtimes \gamma(N')$. This yields a natural homomorphism

$$\pi\colon U_N \to U_N/N \cong (N \rtimes A)/(N \rtimes \gamma(N')) \cong T_N$$

with kernel N.

(iii) As in (ii), $(N' \times 1)D = N' \rtimes \gamma(N')$. Since $\gamma(N') \leq \mathrm{Inn}(N)$ acts trivially on N/N', T_N acts on N/N'. This yields a natural isomorphism

$$U_N/N' \cong (N \rtimes A)/(N' \rtimes \gamma(N')) \cong (N/N') \rtimes T_N.$$

In particular, N/N' has a complement in U_N/N'.

(iv) Now let $N \trianglelefteq B$, and let L/N' be a complement of N/N' in B/N', i.e. $B = NL$ and $N \cap L = N'$. Since $N \trianglelefteq B$, conjugation extends γ to a homomorphism $\gamma\colon B \to A$, $\gamma_b(x) := bxb^{-1}$ for $x \in N$. Letting L act on N through γ, the multiplication map $\mu\colon N \rtimes L \to B$, $(n, l) \mapsto nl$, is a homomorphism:

$$\mu\big((n,l)(m,k)\big) = \mu\big((nlml^{-1}, lk)\big) = nlmk = \mu(n,l)\mu(m,k).$$

It is surjective because $B = NL$. For $(n,l) \in \ker\mu$ we have $l = n^{-1} \in N \cap L = N'$, so $\ker\mu = \{(d^{-1}, d) : d \in N'\}$. On the other hand, $N \rtimes L \to N \rtimes A$, $(n,l) \mapsto (n, \gamma_l)$ is a homomorphism mapping $\ker\mu$ into D, by the very definition of D. Hence the composition $N \rtimes L \to N \rtimes A \to U_N$ is trivial on $\ker\mu$ and factors through μ, yielding a homomorphism $f\colon B \to U_N$ with $f(nl) = (n, \gamma_l)D$ for $n \in N$ and $l \in L$. In particular, $f(n) = (n,1)D$ for $n \in N$, so f restricts on N to the identification fixed above. The epimorphism $\pi\colon U_N \to U_N/N \cong T_N$ from (ii) induces a homomorphism $\overline{f}\colon B/N \to T_N$ with $\overline{f}(bN) = \pi(f(b))$.

Consider the fiber product

$$P := U_N \times_{T_N} (B/N) = \{(u,x) \in U_N \times (B/N) : \pi(u) = \overline{f}(x)\}$$

and the homomorphism $g\colon B \to P$, $b \mapsto (f(b), bN)$, which lands in P by the definition of $\overline{f}$. If $g(b) = 1$, then $b \in N$ and $f(b) = 1$, so $b = 1$ because f is injective on N; hence g is injective. Since π is surjective with kernel N, $|P| = |N|\,|B/N| = |B|$. Hence g is an isomorphism carrying N onto $N \times 1$, the kernel of the projection onto B/N. $\square$

We can now prove the main theorem stated in the introduction.

Proof of Theorem A. By [3, Theorem 12], Gaschütz's theorem fails for N if $\mathrm{Z}(N) \cap N' \neq 1$. Assume therefore that $\mathrm{Z}(N) \cap N' = 1$, so that Theorem 3.1 applies.

Suppose first that N has a complement K in U_N. Let $N \trianglelefteq B$ such that N/N' has a complement in B/N'. By Theorem 3.1(iv) we may assume that $B = P := U_N \times_{T_N} (B/N)$ and $N = N \times 1$. The subgroup

$$M := P \cap \big(K \times (B/N)\big) \leq P$$

intersects $N \times 1$ trivially, because $K \cap N = 1$. Moreover, let $(u,x) \in P$ and write $u = nk$ with $n \in N$ and $k \in K$. Since $\ker\pi = N$, we get $\pi(k) = \pi(u)$ and therefore $(k,x) \in P$, that is, $(k,x) \in M$. Now $(u,x) = (n,1)(k,x) \in (N \times 1)M$. Hence M is a complement of N in P. This verifies condition (ii) of Theorem 2.1, so $\mathcal{G}(N)$ holds.

Suppose conversely that N has no complement in U_N. By Theorem 3.1(iii), N/N' has a complement in U_N/N', so condition (ii) of Theorem 2.1 fails for $B = U_N$. By that theorem, $\mathcal{G}(N)$ fails. $\square$

For an abelian group N, we have $N' = 1$, so $U_N = \mathrm{Hol}(N)$ splits over N. At the other extreme, suppose that N is perfect and $\mathrm{Z}(N) = 1$. The map $\mathrm{Hol}(N) \to A$, $(n, \alpha) \mapsto \gamma_n\alpha$ is a homomorphism because $\gamma_{\alpha(n)} = \alpha\gamma_n\alpha^{-1}$. It is clearly surjective with kernel D. It therefore induces an isomorphism $U_N \cong A$ carrying N onto $\mathrm{Inn}(N)$. Thus Theorem A includes both Gaschütz's theorem and Sambale's characterization for perfect groups [3, Theorem 15]. The following corollary unifies both conditions.

Corollary 3.2. *Let N be a group satisfying $\mathrm{Z}(N) \cap N' = 1$. If $\gamma(N')$ has a complement in A, then Gaschütz's theorem holds for N.*

Proof. Let C be a complement of $\gamma(N')$ in A, and let

$$\overline{C} \coloneqq (1 \times C)D/D \leq U_N.$$

An element $(1, c)D$ lies in the image of N if and only if $(1, c) \in (N \times 1)D = N \rtimes \gamma(N')$, that is, if and only if $c \in \gamma(N')$. Since $C \cap \gamma(N') = 1$, we get $\overline{C} \cap N = 1$. Moreover, the epimorphism $\pi\colon U_N \to T_N$ of Theorem 3.1(ii) sends $\overline{C}$ to $C\gamma(N')/\gamma(N') = T_N$. As $\ker \pi = N$, this gives $N\overline{C} = U_N$. Hence $\overline{C}$ is a complement of N in U_N, and Gaschütz's theorem holds for N by Theorem A. □

The Frobenius group $N \cong (C_3 \times C_3) \rtimes Q_8$ shows that the condition in Corollary 3.2 is not necessary for $\mathcal{G}(N)$.

4. Solution of a problem of Sambale

In [3], the validity of Gaschütz's theorem was decided for every group of order less than 144, and the first open case

$$N \coloneqq (C_3^2 \rtimes Q_8) \times C_2 = \texttt{SmallGroup}(144, 187)$$

was posed as a problem. A straightforward computation with GAP [1][2] shows that N has no complement in $U_N \coloneqq \mathrm{Hol}(N)/D$ (a group of order 13,824). A witness G of the failure of $\mathcal{G}(N)$ can be constructed via Theorem 2.1 with $B = U_N$ and $p = 5$. This yields

$$|G| = p \cdot |U_N| \cdot |N|^p = 5 \cdot 13{,}824 \cdot 144^5 = 2^{29} \cdot 3^{13} \cdot 5 \approx 4.3 \cdot 10^{15}.$$

The starting point of this paper was a smaller witness of order $2^{25} \cdot 3^2 \cdot 5$, found by GPT.

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[2] See the file attached to this pdf.

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