

On a fixed point formula of Navarro–Rizo

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Abstract

Let G be a π -separable group with a Hall π -subgroup H of order n . For $x \in H$ let $\lambda(x)$ be the number of Hall π -subgroups of G containing x . We show that $\prod_{d|n} \prod_{x \in H} \lambda(x^d)^{\frac{n}{d} \mu(d)} = 1$, where μ is the Möbius function. This generalizes fixed point formulas for coprime actions by Brauer, Wielandt and Navarro–Rizo. We further investigate an additive version of this formula.

Keywords: fixed points, coprime action, Sylow subgroups, p -solvable groups, π -separable groups

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1 Introduction

Navarro and Rizo [5] proved the following fixed point equation related to formulas of Brauer and Wielandt.

Theorem 1 (Navarro–Rizo). *Let P be a finite p -group acting on a p' -group N . Then*

$$|C_N(P)| = \left(\prod_{x \in P} \frac{|C_N(x)|}{|C_N(x^p)|^{1/p}} \right)^{\frac{p}{|P|(p-1)}}. \quad (1.1)$$

For a finite group G and $x \in P \in \text{Syl}_p(G)$ let $\lambda_G(x)$ be the number of Sylow p -subgroups of G containing x . In the situation of Theorem 1, the Sylow p -subgroups of $G := N \rtimes P$ have the form nPn^{-1} for $n \in N$. Note that $N_G(P) = C_N(P)P$. Suppose that $x \in P \cap nPn^{-1}$. Then there exists $y \in P$ such that $x = nyn^{-1}$. Since $[n, y] = xy^{-1} \in P \cap N = 1$, it follows that $x = y$ and $n \in C_N(x)$. Hence, $\lambda_G(x) = |C_N(x) : C_N(P)|$ for all $x \in P$. Now (1.1) turns into the more elegant formula:

$$\prod_{x \in P} \lambda_G(x^p) = \prod_{x \in P} \lambda_G(x)^p. \quad (1.2)$$

We show that this holds more generally for all p -solvable groups. In fact, our main theorem applies to π -separable groups, where π is any set of primes. Recall that a π -separable group G has a unique conjugacy class of Hall π -subgroups. For a π -element $x \in G$ let $\lambda_G(x)$ be the number of Hall π -subgroups of G containing x .

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Theorem 2. *Let G be a π -separable group with a Hall π -subgroup H of order n . Then*

$$\prod_{d|n} \left(\prod_{x \in H} \lambda_G(x^d)^{\frac{n}{d}} \right)^{\mu(d)} = 1, \quad (1.3)$$

where μ is the Möbius function.

For $\pi = \{p\}$, (1.3) becomes (1.2). We will show in Proposition 4 that (1.3) holds for arbitrary groups whenever they have a cyclic Hall π -subgroup (by a result of Wielandt, the Hall π -subgroups are conjugate in this situation too, see [2, Satz III.5.8]). In general, the left hand side of (1.2) can be larger or smaller than the right hand side (consider $G = A_5$ and $G = \text{GL}(3, 2)$ for $p = 2$). We did not find a non-solvable group fulfilling (1.2) for $p = 2$.

In the last section we obtain the following additive version of (1.3).

Theorem 3. *Let G be a finite group with a Hall subgroup H of order n . Then*

$$\frac{1}{n^2} \sum_{d|n} \mu(d) \sum_{h \in H} \lambda_G(h^d)^{\frac{n}{d}}$$

is a non-negative integer, which is zero if and only if $1 \neq H \trianglelefteq G$.

2 The proof of Theorem 2

In the first step we reduce Theorem 2 to π -nilpotent groups. Let $\alpha(G)$ be the left hand side of (1.3). Since

$$\alpha(G) = \left(\prod_{d|n} \prod_{x \in H} \lambda_G(x^d)^{\mu(d)/d} \right)^n,$$

we may replace n by the product of the prime divisors of $|H|$. Suppose that $p \in \pi$ does not divide n . Then $\langle x^d \rangle = \langle x^{dp} \rangle$ for all $x \in H$ and $d | n$. Since $\lambda_G(x^d)$ only depends on $\langle x^d \rangle$, it follows that

$$\prod_{d|np} \prod_{x \in H} \lambda_G(x^d)^{\frac{np}{d} \mu(d)} = \prod_{d|n} \prod_{x \in H} \lambda_G(x^d)^{\frac{np}{d} \mu(d)} \lambda_G(x^d)^{-\frac{n}{d} \mu(d)} = \alpha(G)^{p-1}.$$

Hence, we can assume that $n = \prod_{p \in \pi} p$.

Suppose that $N := O_\pi(G) \neq 1$. Since N lies in every Hall π -subgroup of G , we have $\lambda_G(xy) = \lambda_G(x)$ for all $x \in H$ and $y \in N$. Moreover, $\lambda_G(x) = \lambda_{G/N}(xN)$. By induction on $|G|$, we obtain

$$\alpha(G) = \prod_{x \in H} \prod_{d|n} \lambda_G(x^d)^{\frac{n}{d} \mu(d)} = \prod_{xN \in H/N} \left(\prod_{d|n} \lambda_{G/N}(x^d N)^{\frac{n}{d} \mu(d)} \right)^{|N|} = \alpha(G/N)^{|N|} = 1.$$

Thus, we may assume that $O_\pi(G) = 1$. Then $N := O_{\pi'}(G) \neq 1$. By the argument from the introduction, we have $\lambda_{HN}(x) = |C_N(x) : C_N(H)|$ for $x \in H$. If x lies in another Hall π -subgroup K , then

$$\lambda_{HN}(x) = |C_N(x) : C_N(H)| = |C_N(x) : C_N(K)| = \lambda_{KN}(x),$$

because H and K are conjugate in G . It follows that $\lambda_G(x) = \lambda_{HN}(x) \lambda_{G/N}(xN)$. Hence, by the same argument as before, $\alpha(G) = \alpha(HN) \alpha(G/N)$. By induction on $|G|$, we may assume that $G = HN$, i.e. G is a π -nilpotent group.

Next, we reduce to the case where H is cyclic. Under this assumption, the result holds for arbitrary groups.

Proposition 4. *Let G be a group with a cyclic Hall subgroup H . Then (1.3) holds.*

Proof. For every generator h of H we have $\lambda_G(h) = 1$. Now let $h \in H$ be of order $\frac{n}{e} < n$. Then for every $d \mid e$ there exist exactly d elements $x \in H$ such that $x^d = h$. Hence, the exponent of $\lambda_G(h)$ in (1.3) is

$$\sum_{d \mid e} d \frac{n}{d} \mu(d) = n \sum_{d \mid e} \mu(d) = 0. \quad \square$$

For the general case, let $\mathcal{Z}$ be the set of cyclic subgroups of H . Instead of running over all elements of H , we run over $Z \in \mathcal{Z}$ and then over $z \in Z$. In order to track multiplicity, we use the Möbius function μ of the lattice $\mathcal{Z}$. Since the subgroups of a cyclic group of order d are in bijection to the divisors of d , we have $\mu(Z, W) = \mu(|W/Z|)$ whenever $Z \leq W$ and $\mu(Z, W) = 0$ otherwise. We define $f(Z) := \sum_{W \in \mathcal{Z}} \mu(Z, W)$. Recall the inversion formula for Euler's totient function:

$$\varphi(n) = \sum_{d \mid n} \frac{n}{d} \mu(d). \quad (2.1)$$

For $W \in \mathcal{Z}$, let $[W]$ be the set of generators of W . For any function $\gamma : G \rightarrow \mathbb{Z}$, we have

$$\prod_{w \in W} \gamma(w) = \prod_{Z \leq W} \prod_{z \in [Z]} \gamma(z).$$

By Möbius inversion, it follows that $\prod_{w \in [W]} \gamma(w) = \prod_{Z \leq W} \prod_{z \in Z} \gamma(z)^{\mu(Z, W)}$ and

$$\prod_{x \in H} \gamma(x) = \prod_{W \in \mathcal{Z}} \prod_{w \in [W]} \gamma(w) = \prod_{W \in \mathcal{Z}} \prod_{Z \leq W} \prod_{z \in Z} \gamma(z)^{\mu(Z, W)} = \prod_{Z \in \mathcal{Z}} \left(\prod_{z \in Z} \gamma(z) \right)^{f(Z)}. \quad (2.2)$$

Counting the number of factors on both sides also reveals that

$$|H| = \sum_{Z \in \mathcal{Z}} |Z| f(Z). \quad (2.3)$$

Assuming $G = NH$, we have

$$\lambda_G(x) = |C_N(x) : C_N(H)| = |C_N(x) : C_N(Z)| |C_N(Z) : C_N(H)| = \lambda_{ZN}(x) |C_N(Z) : C_N(H)| \quad (2.4)$$

for $x \in Z \in \mathcal{Z}$. Now we can put everything together and apply Proposition 4:

$$\begin{aligned} \alpha(G) &\stackrel{(2.2)}{=} \prod_{Z \in \mathcal{Z}} \left(\prod_{z \in Z} \prod_{d \mid n} \lambda_G(z^d)^{\frac{n}{d} \mu(d)} \right)^{f(Z)} \stackrel{(2.4)}{=} \prod_{Z \in \mathcal{Z}} \left(\alpha(NZ) \prod_{d \mid n} |C_N(Z) : C_N(H)|^{|Z| \frac{n}{d} \mu(d)} \right)^{f(Z)} \\ &\stackrel{(2.1)}{=} \prod_{Z \in \mathcal{Z}} |C_N(Z) : C_N(H)|^{\varphi(n) |Z| f(Z)} \stackrel{(2.3)}{=} \left(|C_N(H)|^{-|H|} \prod_{Z \in \mathcal{Z}} |C_N(Z)|^{|Z| f(Z)} \right)^{\varphi(n)}. \end{aligned}$$

At this point, the claim follows from Wielandt's formula [8, Satz 2.3], which we prove for sake of self-containment.

Theorem 5 (Wielandt). *Let H be a group acting coprimely on a group N . Then*

$$|C_N(H)|^{|H|} = \prod_{Z \in \mathcal{Z}} |C_N(Z)|^{|Z| f(Z)}.$$

Proof. Since Wielandt's paper is hard to follow (even for a German native speaker), we use some modern ingredients. We consider N as a H -set via conjugation. By a theorem of Hartley–Turull [1, Lemma 2.6.2] (see also [7, Satz 9.20]), there exist a direct product A of elementary abelian groups and an isomorphism of H -sets $\varphi : N \rightarrow A$, i. e. $\varphi(n^h) = \varphi(n)^h$ for $n \in N$ and $h \in H$. It follows that $\varphi(C_N(Z)) = C_A(Z)$ for $Z \in \mathcal{Z}$. Thus, we may replace N by A . Then N decomposes into its (characteristic) Sylow subgroups $N = N_1 \times \dots \times N_k$. Since $C_N(Z) = C_{N_1}(Z) \times \dots \times C_{N_k}(Z)$, we may assume further that $N = N_1$ is elementary abelian. Thus, N is an $\mathbb{F}_p H$ -module for some prime p not dividing $|H|$. The corresponding Brauer character $\chi : H \rightarrow \mathbb{C}$ can be regarded as an ordinary character since $|H|$ is coprime to p . We further extend χ to the complex group algebra $\mathbb{C}H$. For $S \subseteq H$ let $S^+ := \sum_{s \in S} s \in \mathbb{C}H$. The additive version of (2.2) reads

$$H^+ = \sum_{Z \in \mathcal{Z}} f(Z) Z^+.$$

By the first orthogonality relation, $\chi(Z^+) = |Z|[\chi_Z, 1_Z]$, where $[\chi_Z, 1_Z]$ is the multiplicity of the trivial character 1_Z as a constituent of the restriction χ_Z . On the other hand, we have $|C_N(Z)| = p^{[\chi_Z, 1_Z]}$. It follows that

$$|C_N(H)|^{|H|} = p^{|H|[\chi, 1_H]} = p^{\chi(H^+)} = p^{\sum_{Z \in \mathcal{Z}} |Z|[\chi_Z, 1_Z] f(Z)} = \prod_{Z \in \mathcal{Z}} |C_N(Z)|^{|Z| f(Z)}. \quad \square$$

The proof of Theorem 5 relies on the Feit–Thompson theorem to guarantee that H or N is solvable. In comparison, the proof of Theorem 1 does not require representation theory, but uses the fact that Sylow subgroups are nilpotent.

3 The proof of Theorem 3

A π -element $x \in G$ lies in a Hall π -subgroup H if and only if $x \in N_G(H)$. Hence, the map $\lambda_G : H \rightarrow \mathbb{Z}$ is the permutation character of the conjugation action of H on the set $\text{Hall}_\pi(G)$ of all Hall π -subgroups of G (we do not assume that these subgroups are conjugate in G). We use the following recipe to construct a related character.

Theorem 6. *Let χ be a character of a finite group H and let α be a character of a permutation group $A \leq S_n$. For $a \in A$ let $c_i(a)$ be the number of cycles of a of length i . Then the map $\chi_\alpha : H \rightarrow \mathbb{C}$ with*

$$\chi_\alpha(h) = \frac{1}{|A|} \sum_{a \in A} \alpha(a) \prod_{i=1}^n \chi(h^i)^{c_i(a)}$$

for $h \in H$ is a character or the zero map.

Proof. See [3, Theorem 7.7.7 and Eq. (7.7.9)]. $\square$

We assume the notation of Theorem 3 and choose a cyclic subgroup $A \leq S_n$ generated by a cycle of length n . Let $\alpha \in \text{Irr}(A)$ be a faithful character. For $B \leq A$ of order d , $\sum_{b \in [B]} \alpha(b)$ is the sum of the primitive roots of unity of order d . A simple Möbius inversion shows that this sum equals $\mu(d)$. Moreover, every $b \in [B]$ is a product of n/d disjoint cycles of length d . Thus, $c_d(b) = n/d$ and $c_i(b) = 0$ for $i \neq d$.

We apply Theorem 6 with $\chi = \lambda_G$. For $h \in H$ we compute

$$\chi_\alpha(h) = \frac{1}{n} \sum_{d|n} \sum_{\substack{a \in A \\ |\langle a \rangle| = d}} \alpha(a) \lambda_G(h^d)^{\frac{n}{d}} = \frac{1}{n} \sum_{d|n} \mu(d) \lambda_G(h^d)^{\frac{n}{d}}.$$

Taking the scalar product of χ_α and the trivial character of H , shows that

$$\beta_G(H) := \frac{1}{n^2} \sum_{h \in H} \sum_{d|n} \mu(d) \lambda_G(h^d)^{\frac{n}{d}}$$

is a non-negative integer. This confirms the first part of Theorem 3.

If $H \trianglelefteq G$, then $\lambda_G(h) = 1$ for all $h \in H$ and it follows that $\beta_G(H) = 0$ unless $H = 1$ (where $\beta_G(H) = 1$). Now assume that H is not normal in G . In particular, $n > 1$. Let $t := |\text{Hall}_\pi(G)|$. If $t = 2$, then $|G : N_G(H)| = 2$ and $N_G(H) \trianglelefteq G$. But this would imply that $H^g = O_\pi(N_G(H)) = H$ for every $g \in G$. Hence, $t \geq 3$.

Next we investigate the contribution of $d = 1$ to $\beta_G(H)$. Note that $\lambda_G(h)^n$ is the number of fixed points of h on $\text{Hall}_\pi(G)^n$, acting diagonally. The number of orbits of H on $\text{Hall}_\pi(G)$ is at most t^n/n . Using Burnside's lemma and the trivial estimate $\lambda_G(h^d) \leq t$ for $d \neq 1$ and $h \in H$, we obtain

$$n\beta_G(H) \geq \frac{1}{n} \sum_{h \in H} \lambda_G(h)^n - \frac{1}{n} \sum_{h \in H} \sum_{1 \neq d|n} t^{\frac{n}{d}} \geq \frac{t^n}{n} - \sum_{1 \neq d|n} t^{\frac{n}{d}}.$$

It suffices to show that

$$n \sum_{1 \neq d|n} t^{\frac{n}{d}} < t^n.$$

If n is a prime, this reduces to $nt < t^n$ and we are done as $t \geq 3$. If $n = 4$ and $t = 3$, the claim can be checked directly. In all other cases, one can verify that $n \leq t^{\frac{n}{2}-1}$ and

$$n \sum_{1 \neq d|n} t^{\frac{n}{d}} \leq \sum_{k=0}^{n-1} t^k = \frac{t^n - 1}{t - 1} < t^n.$$

This finishes the proof.

We remark that the degree $\chi_\alpha(1) = \frac{1}{n} \sum_{d|n} \mu(d) t^{n/d}$ has several interesting interpretations. For instance, if t is a prime power, then $\chi_\alpha(1)$ is the number of irreducible polynomials of degree n over the finite field $\mathbb{F}_t$ (see [6, Corollary 10.2.3]). In general, $\chi_\alpha(1)$ is the rank of the n -th quotient of the lower central series of a free group of rank t (see [4, Theorem 5.11 and Corollary 5.12]).

We now give an interpretation of $\beta_G(H)$ in a special case. Suppose that H is nilpotent with regular Sylow subgroups (for all primes). Then the d -powers form a subgroup H^d , and for every $h \in H^d$ there exist exactly $|H : H^d|$ elements $x \in H$ with $x^d = h$ (see [2, Hauptsatz III.10.5 and Satz III.10.6]). Burnside's lemma applied to H^d acting on $\text{Hall}_\pi(G)^{n/d}$, yields

$$\beta_G(H) = \frac{1}{n} \sum_{d|n} \mu(d) f_{n/d}(H^d),$$

where $f_{n/d}(H^d)$ is the number of orbits of H^d on $\text{Hall}_\pi(G)^{n/d}$.

Finally, we comment on a curiosity. In order to produce similar quantities as $\beta_G(H)$, we may replace λ_G by any character of any finite group G . For instance, let τ be the conjugation character of $G = A_5$ on $\text{Syl}_3(G)$. We compute

[illegible]

Similar curious numbers arise from $G \in \{S_5, \text{PSL}(2, 9), \text{PGL}(2, 9)\}$. This can be explained by the presents of large powers of $\tau(1) = 10$.

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