# A counterexample to a conjecture of A. R. Miller

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**Abstract**

Let $\chi$ be an irreducible character of a finite group $G$. A. R. Miller conjectured that the proportion of elements $g \in G$ such that $\chi(g)$ is zero or a root of unity is at least $1/2$. We construct a character of a perfect group of order 69,120 such that this proportion is $511/1152 \approx 0.44$.

**Keywords:** character values, Schur–Siegel problem
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Let $x \in \mathbb{R} \setminus \{1\}$ be an algebraic integer such that all Galois conjugates $x_1, \dots, x_n$ of $x$ are real and positive. A well-known theorem of Siegel [9, Theorem III] states that

$$\overline{\mathrm{tr}}(x) := \frac{x_1 + \ldots + x_n}{n} \geq \frac{3}{2}$$

with equality if and only if $x = \frac{3 \pm \sqrt{5}}{2}$. Thompson has observed that this theorem applies to $x = |\chi(g)|^2$ for any character $\chi$ of finite groups $G$ and $g \in G$. When $\chi$ is irreducible, the orthogonality relations imply that the proportion of elements $g \in G$ such that $\chi(g)$ is zero or a root of unity is at least $1/3$, i. e.

$$\lambda(\chi) := \big|\{g \in G : \chi(g) = 0 \text{ or } |\chi(g)| = 1\}\big| \geq \frac{|G|}{3}$$

(see [3, Problem 3.15]).

Siegel's theorem has been improved many times, but the exact value of

$$\lambda := \liminf_x \overline{\mathrm{tr}}(x)$$

is still unknown. Most recently, Orloski–Sardari–Smith [8, p. 2010] proved that $\lambda \geq 1.80203$. On the other hand, the sequence $\{4\cos(2\pi/p)^2 : p \text{ prime}\}$ shows that $\lambda \leq 2$.

Since character values are cyclotomic integers (i. e. integers of a cyclotomic field), it is expected that stronger bounds can be given. In fact, Cassels [1, Lemma 3] proved for every cyclotomic integer $x$ that $\overline{\mathrm{tr}}(|x|^2) \geq 2$ unless $x$ is a sum of at most two roots of unity. Perhaps motivated by this result, Miller [4, Conjecture 1] conjectured that $\lambda(\chi) \geq |G|/2$ for all $\chi \in \mathrm{Irr}(G)$ (see also [5, 6]). He proved this bound for all nilpotent groups, among other cases. This was extended to all groups with a Sylow tower by Moretó–Navarro [7, Theorem C]. The conjecture seems to be open for solvable groups.

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Using GAP [2], we found a perfect group $G \cong C_{12}.(C_2^4 \rtimes A_6)$ of order 69,120 with eight (Galois conjugate) irreducible characters $\chi$ of degree 12 such that $\lambda(\chi) = 511/1152 \approx 0.44$. This can be checked with the following code:

```
n:=69120;
G:=PerfectGroup(n,2);
ct:=CharacterTable(G);
chi:=First(Irr(ct), c -> c[1]=12);
pos:=PositionsProperty(chi, x -> x=0 or x*ComplexConjugate(x)=1);
Sum(SizesConjugacyClasses(ct){pos})/n;
```

In accordance with Siegel's and Cassels's theorems mentioned above, there exists $g \in G$ such that $\chi(g) = \zeta_5 + \overline{\zeta_5} = 2\cos(2\pi/5)$ and $|\chi(g)|^2 = \frac{3-\sqrt{5}}{2}$ in this example. In fact, there is no $g \in G$ such that $\chi(g)$ is a root of unity. There are more such examples among perfect groups involving $A_6$, but we did not find a smaller value of $\lambda(\chi)$. It remains an open problem to determine $\min \lambda(\chi)$ over all finite groups.

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## References

[1] J. W. S. Cassels, *On a conjecture of R. M. Robinson about sums of roots of unity*, J. Reine Angew. Math. **238** (1969), 112–131.

[2] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.15.0*; 2025, (`https://www.gap-system.org`).

[3] I. M. Isaacs, *Character theory of finite groups*, AMS Chelsea Publishing, Providence, RI, 2006.

[4] A. R. Miller, *Zeros and roots of unity in character tables*, Enseign. Math. **70** (2024), 151–164.

[5] A. R. Miller, *Short character values on large conjugacy classes*, arXiv:2411.19283.

[6] A. R. Miller, *Denseness results for zeros and roots of unity in character tables*, arXiv:2507.15153.

[7] A. Moretó and G. Navarro, *p-groups and zeros of characters*, Arch. Math. (Basel) **121** (2023), 343–350.

[8] B. J. Orloski, N. T. Sardari and A. Smith, *New lower bounds for the Schur-Siegel-Smyth trace problem*, Math. Comp. **94** (2025), 2005–2040.

[9] C. L. Siegel, *The trace of totally positive and real algebraic integers*, Ann. of Math. (2) **46** (1945), 302–312.