

Characterizing inner automorphisms and realizing outer automorphisms

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Abstract

We give elementary proofs of the following two theorems on automorphisms of a finite group G : (1) An automorphism of G is inner if and only if it extends to an automorphism of every finite group containing G . (2) There exists a finite group, whose outer automorphism group is isomorphic to G . The first theorem was proved by Pettet using a graph-theoretical construction of Heineken–Liebeck. A Lie-theoretical proof of the second theorem was sketched by Cornulier in a MathOverflow post. Our proofs are purely group-theoretical.

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1 Introduction

An automorphism α of a group G is called *inner* if there exists some $g \in G$ such that $\alpha(x) = gxg^{-1}$ for all $x \in G$. If G is a subgroup of a group H , it is clear that g still induces an (inner) automorphism of H . In 1987, Schupp [27] has shown conversely that inner automorphisms are characterized by this property, i. e. if $\alpha \in \text{Aut}(G)$ extends to every group containing G , then α is inner. According to [8], this has answered a question of Macintyre. The question was asked again much later by Bergman [1, p. 93], who obtained a partial answer in the language of category theory. Using free products and small cancellation theory, Schupp constructs for a non-inner $\alpha \in \text{Aut}(G)$ an infinite group H such that α does not extend to H (if G is countable, the construction is already contained in Miller–Schupp [23]). One may ask whether inner automorphisms of *finite* groups G are characterized by the property that they extend to all *finite* groups containing G .

The first step in this direction was a paper from 1974 of Heineken–Liebeck [13], who constructed a finite p -group P such that the image of the canonical map $\text{Aut}(P) \rightarrow \text{Aut}(P/\mathbf{Z}(P))$ is isomorphic to G . Their construction relies on a variation of Frucht’s theorem on the automorphism group of graphs, and requires a treatment of special cases. Using more advanced graph theoretical theorems, Lawton [20] came up with a shorter proof. Subsequently, Webb [28] has refined the construction (again at the cost of more graph theory) to obtain a special p -group P (that means $P' = \mathbf{Z}(P) = \Phi(P)$ is elementary abelian) with the desired property (see also Hughes [14]). Only after Schupp’s paper, Pettet [24] noticed in 1989 that this result implies Schupp’s theorem for finite groups (and more restricted families of groups).

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In the same paper, he also obtained the dual statement for factor groups (see the theorem below). Eventually, Pettet [25] gave a new proof of Schupp’s original theorem with the same graph theoretical approach. An alternative construction of P was established by Bryant–Kovács [4] in 1978 making use of Lie theory (see also Huppert–Blackburn [16, Theorem VIII.13.5] and Hartley–Robinson [12]).

The first objective of this paper is to provide a new elementary proof of the following theorem, which avoids small cancellation theory, graph theory and Lie theory.

Theorem (PETTET). *For every automorphism α of a finite group G the following statements are equivalent:*

- (1) α is an inner automorphism.
- (2) α extends to every finite group containing G .
- (3) α lifts to every finite group \hat{G} such that $\hat{G}/N \cong G$ for some characteristic subgroup $N \leq \hat{G}$.

The strategy is to replace the p -group P in [13] by a semidirect product $N = Q \rtimes P$, where Q is an elementary abelian q -group and P is a p -group of nilpotency class 2. In contrast to the papers cited above, we do not make an effort to minimize N .

Our second objective concerns the inverse problem on automorphism groups. By a theorem of Ledermann–Neumann [21], there exist only finitely many finite groups with a given automorphism group (for an elementary proof see [26]). However, not every group actually occurs as an automorphism group. For instance, it is a popular (and easy) exercise that a non-trivial cyclic group of odd order cannot be an automorphism group. The situation changes if we instead consider the *outer* automorphism group $\text{Out}(H) := \text{Aut}(H)/\text{Inn}(H)$ of groups H . Indeed, Matumoto [22] proved that for every group G there exists a group H such that $\text{Out}(H) \cong G$ (for finite G this was established earlier by Kojima [19]). Similar to Schupp’s paper, the construction of H is based on an HNN-extension and yields infinite groups. In later work, it was shown that H can be chosen to be locally finite, finitely generated (if G is countable), metabelian or simple (see [3, 5, 11, 7]). Problem 16.59 in the Kourovka Notebook [18] has asked if H can be chosen finite when G is finite. This was answered in 2020 by Cornulier on MathOverflow [6].

Theorem (CORNULIER). *Every finite group is the outer automorphism group of some finite group.*

His proof uses Lie algebras and is not easy to follow as it is written backwards. In Section 3 we present a purely group-theoretical proof based on Cornulier’s ideas. As in Pettet’s theorem, the idea is to construct a semidirect product $P \rtimes Q$, but this time Q is abelian and P is a p -group of exponent p with a certain nilpotency class. Our group is slightly smaller compared to Cornulier’s construction.

The construction of our group has already been used by Entin–Tsang [9] to prove that every finite group is isomorphic to the normalizer quotient of a subgroup of a symmetric group.

2 Characterizing inner automorphisms

Our first lemma is a prototype of Theorem 4.

Lemma 1. *Let G be a group acting faithfully on a cyclic group N . Then every automorphism of $\hat{G} := N \rtimes G$ normalizing N , centralizes \hat{G}/N .*

Proof. We identify $N = \langle x \rangle$ and G with the natural subgroups of \hat{G} . Let $\alpha \in \text{Aut}(\hat{G})$ normalizing N . For a fixed $g \in G$ there exist $s, t \in \mathbb{Z}$ with $gxg^{-1} = x^s$ and $\alpha(x) = x^t$. Hence,

$$gx^t g^{-1} = x^{st} = \alpha(x)^s = \alpha(x^s) = \alpha(gxg^{-1}) = \alpha(g)x^t\alpha(g)^{-1}.$$

Since G acts faithfully on $N = \langle x^t \rangle$, we obtain $\alpha(g) \equiv g \pmod{N}$ as desired. \square

In the following we develop some elementary facts of finitely presented groups. For elements x_1, x_2, \dots of a group G we define commutators by $[x_1, x_2] = x_1x_2x_1^{-1}x_2^{-1}$ and $[x_1, \dots, x_k] := [x_1, [x_2, \dots, x_k]]$ for $k \geq 3$. The commutator subgroup of G is denoted by G' .

Lemma 2. *Let p be a prime and $a, b \in \mathbb{Z}$. Then*

$$P := \langle x, y \mid [x, x, y] = [y, x, y] = 1, x^p = [x, y]^a, y^p = [x, y]^b \rangle$$

is a non-abelian group of order p^3 .

Proof. The commutator relations show that P has nilpotency class ≤ 2 , i. e. $P' \leq Z(P)$. Hence, $[x, y]^p = [x^p, y] = 1$ and $|P'| \leq p$ (see [15, Hilfssatz III.1.3]). It follows that $|P| \leq p^3$. To complete the proof, we construct a group of order p^3 realizing the given relations. Suppose first that $p = 2$. If ab is even, then $P \cong D_8$ and otherwise $P \cong Q_8$. Thus, let $p > 2$. If $a \equiv b \equiv 0 \pmod{p}$, then the extraspecial group of exponent p fulfills the relations as is well-known. Now let $p \nmid a$ and $a' \in \mathbb{Z}$ such that $aa' \equiv -b \pmod{p}$. For $y' := x^{a'}y$ we compute

$$(y')^p = x^{pa'}y^p[y, x^{a'}]^{\binom{p}{2}} = x^{pa'}y^p = [x, y]^{aa'+b} = 1$$

and $[x, y'] = [x, y]$ by [15, Hilfssatz III.1.3]. Hence, replacing y by y' leads to $b = 0$. This remains true when we replace y by y^{-a} . Then $x^p = [x, y]^{-1} = [y, x]$ and

$$P \cong \langle x, y \mid x^{p^2} = y^p = 1, yxy^{-1} = x^{1+p} \rangle \cong C_{p^2} \times C_p. \quad \square$$

Lemma 3. *Let F be the free group in the free generators x_1, \dots, x_n . Let $c_1, \dots, c_n \in F'$. For a prime p , let P be the group generated by x_1, \dots, x_n subject to the relations $x_i^p = c_i$ and $[x_i, x_j, x_k] = 1$ for all $1 \leq i, j, k \leq n$. Then P/P' is an elementary abelian p -group with basis $\{x_iP' : i = 1, \dots, n\}$ and P' is an elementary abelian p -group with basis $\{[x_i, x_j] : 1 \leq i < j \leq n\}$. In particular, $|P| = p^{\binom{n+1}{2}}$.*

Proof. It follows from $x_i^p = c_i \in P'$ that P/P' is an elementary abelian p -group generated by $\{x_iP' : i = 1, \dots, n\}$. Since $[x_i, x_j, x_k] = 1$ for all i, j, k , P has nilpotency class ≤ 2 . Hence, $[xy, z] = [x, z][y, z]$ and $[x, yz] = [x, y][x, z]$ for all $x, y, z \in P$ (see [15, Hilfssatz III.1.2]). In particular, $[x_i, x_j]^p = [x_i^p, x_j] = [c_i, x_j] = 1$. This shows that P' is an elementary abelian p -group generated by $\{[x_i, x_j] : 1 \leq i < j \leq n\}$.

Suppose that $x := \prod_{i < j} [x_i, x_j]^{a_{ij}} = 1$ for some integers $0 \leq a_{ij} \leq p - 1$. We fix $i < j$ and consider the free group F_2 generated by y_i and y_j . Let $\varphi : F \rightarrow F_2$ be the homomorphism defined by

$$\varphi(x_k) = \begin{cases} y_k & \text{if } k \in \{i, j\}, \\ 1 & \text{otherwise} \end{cases}$$

for $k = 1, \dots, n$. Set

$$P_2 := \langle y_i, y_j \mid [y_i, y_i, y_j] = [y_j, y_i, y_j] = 1, y_i^p = \varphi(c_i), y_j^p = \varphi(c_j) \rangle.$$

As elements of P_2 , $\varphi(c_i)$ and $\varphi(c_j)$ are (possibly trivial) powers of $[y_i, y_j]$. Thus by Lemma 2, P_2 is a non-abelian group of order p^3 . Since every relation of P in the x_k is satisfied by a relation of P_2 in the y_k , φ factors through a homomorphism $\bar{\varphi} : P \rightarrow P_2$. It follows that $[y_i, y_j]^{a_{ij}} = \bar{\varphi}(x) = 1$ and $a_{ij} = 0$. This shows that the commutators $[x_i, x_j]$ with $i < j$ are linearly independent, so they form a basis of P' . Now let $x := \prod_{i=1}^n x_i^{a_i} \in P'$ for some $0 \leq a_i \leq p-1$. Then

$$1 = [x_1, x] = \prod_{i=2}^n [x_1, x_i]^{a_i}$$

and $a_i = 0$ for $i = 2, \dots, n$. Similarly, we obtain $a_1 = 0$. Therefore, P/P' has rank n as claimed. Finally, $|P| = |P'| |P/P'| = p^{\binom{n}{2} + n} = p^{\binom{n+1}{2}}$. \square

Theorem 4. *For every finite group G there exist primes $q > p > |G|$ and a finite $\{p, q\}$ -group N such that every automorphism of $\hat{G} := N \rtimes G$ induces an inner automorphism of $\hat{G}/N \cong G$.*

Proof. Without loss of generality, we assume that $G \neq 1$. Let $p > |G|$ be a prime. Let $x_1, \dots, x_n \in G$ be a generating set of G not containing 1. Let P be the p -group with generators $\{v_g : g \in G\}$ and relations

$$[v_g, v_h, v_k] = 1, \quad v_g^p = \prod_{i=1}^n [v_g, v_{gx_i}]^i$$

for all $g, h, k \in G$. By Lemma 3, P/P' is elementary abelian of rank $|G|$ and P' is elementary abelian with basis $[v_g, v_h]$ where $g < h$ for some fixed total order on G . For a fixed $g \in G$, the elements $\{v_{gh} : h \in G\}$ fulfill the same relations as the v_h . Thus, there exists an automorphism $\varphi_g \in \text{Aut}(P)$ with $\varphi_g(v_h) = v_{gh}$ for all $h \in G$. This gives rise to a regular action $\varphi : G \rightarrow \text{Aut}(P)$, $g \mapsto \varphi_g$.

By an elementary special case of Dirichlet's theorem (see [10, Theorem 3.1.12]), there exists a prime q such that $p \mid q-1$. Let Q be the elementary abelian q -group with basis $\{w_g : g \in G\}$. Let $w_g \mapsto w_g^\zeta$ be an automorphism of order p of $\langle w_g \rangle$. For $g \in G$, define $\gamma_g \in \text{Aut}(Q)$ by

$$\gamma_g(w_h) := \begin{cases} w_g^\zeta & \text{if } h = g, \\ w_h & \text{if } h \neq g. \end{cases} \quad (h \in G)$$

Let $\gamma : P \rightarrow \text{Aut}(Q)$, $v_g \mapsto \gamma_g$ be the homomorphism with kernel P' . This gives rise to the semidirect product $N := Q \rtimes P$ with $Z(N) = P'$. As usual, we identify P and Q with the natural subgroups of N . Then $v_g w_h v_g^{-1} = \gamma_g(w_h)$ for all $g, h \in G$. Again, we have a regular action $\psi : G \rightarrow \text{Aut}(Q)$, $g \mapsto \psi_g$ with $\psi_g(w_h) = w_{gh}$. Moreover, φ and ψ are compatible in the sense that

$$\varphi_g(v_h) \psi_g(w_k) \varphi_g(v_h)^{-1} = v_{gh} w_{gk} v_{gh}^{-1} = \gamma_{gh}(w_{gk}) = \psi_g(\gamma_h(w_k)) = \psi_g(v_h w_k v_h^{-1})$$

for all $g, h, k \in G$. In this way, G acts faithfully on N . As before, we identify G and N with subgroups of $\hat{G} := N \rtimes G$. Then $g v_h g^{-1} = v_{gh}$ and $g w_h g^{-1} = w_{gh}$ for $g, h \in G$.

Now let $\alpha \in \text{Aut}(\hat{G})$. Since $q > p > |G|$, α normalizes the normal Sylow q -subgroup Q , the normal Hall subgroup N , and in turn $Z(N) = P'$. By the Schur–Zassenhaus theorem, there exists some $y \in N$ such that $\alpha(G) = y G y^{-1}$ (we do not require the Feit–Thompson theorem, because N is solvable). Since y centralizes \hat{G}/N , we may compose α with the inner automorphism induced by y^{-1} . Then α normalizes G . Next, we consider the action of α on

$$N/P' \cong \prod_{g \in G} \langle w_g, v_g \rangle \cong (C_q \rtimes C_p)^{|G|}.$$

It follows from

$$|C_{N/P'}(\alpha(v_1)P')| = |C_{N/P'}(v_1P')| = p^{|G|}q^{|G|-1}$$

that $\alpha(v_1) \in \langle v_g, w_g \rangle P'$ for some $g \in G$. Composing α with the inner automorphism induced by g^{-1} , we may assume that $g = 1$. Then α induces an automorphism of $\langle v_1, w_1 \rangle P'/P' \cong C_q \times C_p$. By Lemma 1, there exists $t_1 \in P'$ such that $\alpha(v_1) \equiv v_1 t_1 \pmod{Q}$. Moreover, for every $g \in G$ there exists $t_g \in P'$ with

$$\alpha(v_g) = \alpha(gv_1g^{-1}) \equiv \alpha(g)v_1t_1\alpha(g)^{-1} \equiv v_{\alpha(g)}t_g \pmod{Q}.$$

Consequently,

$$\prod_{i=1}^n [v_1, v_{x_i}]^i = v_1^p = (v_1 t_1)^p \equiv \alpha(v_1)^p \equiv \prod_{i=1}^n [v_1 t_1, v_{\alpha(x_i)} t_{x_i}]^i \equiv \prod_{i=1}^n [v_1, v_{\alpha(x_i)}]^i \pmod{Q}.$$

This is a relation in the linearly independent generators $[v_1, v_g]$ of the elementary abelian group P' . Notice that $\alpha(x_i) \neq 1$ and $n < |G| < p$. Comparing exponents reveals $\alpha(x_i) = x_i$ for $i = 1, \dots, n$. Since $G = \langle x_1, \dots, x_n \rangle$, α induces the identity automorphism on G . \square

How does the group P in the proof above relate to Heineken–Liebeck’s construction? Recall that Frucht’s theorem states that every finite group G is the automorphism group of some finite graph \mathcal{G} . Frucht’s graph is based on the Cayley color graph \mathcal{C} , which depends on a generating set x_1, \dots, x_n of G . More precisely, the vertex set of \mathcal{C} is $\{v_g : g \in G\}$ and there is an arrow $v_g \rightarrow v_h$ of color i if and only if $h = gx_i$. Our proof of Theorem 4 rests on the elementary fact that the color-preserving automorphism group of \mathcal{C} is isomorphic to G . The purpose of the group Q is to enforce automorphisms to permute the generators of P .

Proof of Pettet’s theorem. The implications (1) \Rightarrow (2) and (1) \Rightarrow (3) are obvious. Conversely, if α fulfills (2) or (3), then α extends/lifts to the group \hat{G} constructed in Theorem 4. The theorem implies that α is inner. \square

If G is solvable, the proof above shows that (2) and (3) of Pettet’s theorem can be restricted to solvable extension groups \hat{G} .

3 Realizing outer automorphisms

We start by proving [6, Lemma 2]. Let $n \in \mathbb{N}$ and $p > n$ be a prime. Let $C_p \rtimes C_{p-1}$ be the holomorph of C_p , i. e. C_{p-1} acts faithfully on C_p . Let S_n be the symmetric group of degree n .

Lemma 5. *We have $\text{Out}((C_p \rtimes C_{p-1})^n) \cong S_n$.*

Proof. The argument is similar as for the group $(C_q \rtimes C_p)^n$ in the proof of Theorem 4, although the statement requires the factor $p - 1$. Let $P = \langle x_1, \dots, x_n \rangle \cong C_p^n$ and $Q := \langle y_1, \dots, y_n \rangle \cong C_{p-1}^n$ such that each y_i induces an automorphism of $\langle x_i \rangle$ of order $p - 1$ and centralizes x_j for $j \neq i$. It is obvious that every permutation $\pi \in S_n$ induces an automorphism α_π of $G := P \rtimes Q$ by permuting the factors $\langle x_i, y_i \rangle$. If $\pi \neq \text{id}$, then α_π is not inner, because it acts non-trivially on the abelian quotient $G/P \cong Q$. Hence, S_n induces a subgroup of $\text{Out}(G)$.

Conversely, let $\alpha \in \text{Aut}(G)$ be arbitrary. Then α normalizes the normal Sylow p -subgroup P of G . By the Schur–Zassenhaus theorem, we can further assume that $\alpha(Q) = Q$. Since

$$|C_G(\alpha(x_i))| = |C_G(x_i)| = p^n(p-1)^{n-1},$$

there exists some $\pi \in S_n$ with $\alpha(x_i) \in \langle x_{\pi(i)} \rangle$ for $i = 1, \dots, n$. By composing α with α_π^{-1} , we may assume that $\pi = \text{id}$. A similar argument yields $\alpha(y_i) \in \langle y_i \rangle$. By Lemma 1 applied to $\langle x_i \rangle \rtimes \langle y_i \rangle$, we have $\alpha(y_i) = y_i$. Since $\text{Aut}(\langle x_i \rangle) \cong \langle y_i \rangle$, the action of α on $\langle x_i \rangle$ is induced by conjugation with some power of y_i . Composing α with the corresponding inner automorphism, gives $\alpha(x_i) = x_i$ (this will not affect the action of α on $\langle x_j, y_j \rangle$ for $j \neq i$). Doing this for $i = 1, \dots, n$, leads to $\alpha = \text{id}$. \square

Let F be the free group in the free generators x_1, \dots, x_n . Let $F^p = \langle x^p : x \in F \rangle^F \trianglelefteq F$ be the normal closure of the set of all p -powers in F . Then $\bar{F} := F/F^p$ is the free group of rank n and exponent p . We will identify x_i with its image in \bar{F} . Define the lower central series by $\bar{F}^{[1]} := \bar{F}$ and

$$\bar{F}^{[k+1]} := [\bar{F}, \bar{F}^{[k]}] = \langle [x, y] : x \in \bar{F}, y \in \bar{F}^{[k]} \rangle$$

for $k \geq 1$ as usual. We call $\bar{F}_c := \bar{F}/\bar{F}^{[c+1]}$ the free group of rank n , exponent p and nilpotency class c (we will see in Lemma 7 that the class of \bar{F}_c cannot be smaller than c). The adjective “free” is justified by the following universal property: If G is a nilpotent group of exponent p and class $\leq c$, and if $y_1, \dots, y_n \in G$, then there exists a (unique) homomorphism $\bar{F}_c \rightarrow G$, $x_i \mapsto y_i$ for $i = 1, \dots, n$. We will use this principle frequently in order to verify certain commutator relations in \bar{F}_c .

It is easy to show that $\bar{F}^{[k]}/\bar{F}^{[k+1]}$ is generated by the cosets of the k -fold commutators $[x_{i_1}, \dots, x_{i_k}]$ where $1 \leq i_1, \dots, i_k \leq n$ (see [15, Hilfssatz III.1.11]). It follows that $\bar{F}^{[k]}/\bar{F}^{[k+1]}$ is a finite elementary abelian p -group. In particular, \bar{F}_c is a finite group. The following lemma is certainly known, but I was unable to find a proper reference.

Lemma 6. *For every finite group G of exponent p and $k \in \mathbb{N}$, the map*

$$\begin{aligned} \Phi_k &: (G/G')^k \rightarrow G/G^{[k+1]}, \\ (g_1 G', \dots, g_k G') &\mapsto [g_1, \dots, g_k] G^{[k+1]} \end{aligned}$$

is well-defined and multilinear over \mathbb{F}_p , i.e. for $1 \leq i \leq k$, $h \in G$ and $h' \in G'$,

$$[g_1, \dots, g_{i-1}, g_i h h', g_{i+1}, \dots, g_k] \equiv [g_1, \dots, g_k][g_1, \dots, h, \dots, g_k] \pmod{G^{[k+1]}}.$$

Proof. If we interpret $[x_1]$ as x_1 , then Φ_1 becomes the identity map. Now let $k \geq 2$. Recall that $[G^{[s]}, G^{[t]}] \leq G^{[s+t]}$ by [15, Hauptsatz III.2.11]. A direct computation shows $[xy, z] = [x, y, z][y, z][x, z]$ and $[z, xy] = [xy, z]^{-1} = [z, x][z, y][x, y, z]^{-1}$ for every $x, y, z \in G$. If $i = 1$, we put $z := [g_2, \dots, g_k] \in G^{[k-1]}$ and obtain

$$\begin{aligned} [g_1 h h', g_2, \dots, g_k] &= [g_1 h, h', z][h', z][g_1 h, z] \equiv [g_1 h, z] \equiv [g_1, h, z][h, z][g_1, z] \\ &\equiv [g_1, z][h, z] \equiv [g_1, \dots, g_k][h, g_2, \dots, g_k] \pmod{G^{[k+1]}}. \end{aligned}$$

It remains to consider the component $i \geq 2$. Let $z := [g_{i+1}, \dots, g_k]$. By induction on k , we derive

$$[g_2, \dots, g_i h h', \dots, g_k] \equiv [g_2, \dots, g_i, z][g_2, \dots, h, z] \pmod{G^{[k]}}$$

and

$$[g_1, \dots, g_i h h', \dots, g_k] \equiv [g_1, \dots, g_i, z][g_1, \dots, h, z] \pmod{G^{[k+1]}}. \quad \square$$

The following result resembles [6, Lemma 5].¹

Lemma 7. *Let $n \geq 3$, $\pi \in S_{n-1}$ and $0 \leq a \leq p-1$ such that*

$$[x_1, \dots, x_{n-1}, x_1] \equiv [x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}]^a \pmod{\overline{F}^{[n+1]}}. \quad (3.1)$$

Then $\pi = \text{id}$ and $a = 1$.

Proof. By the universal property, it suffices to prove the claim for any elements x_1, \dots, x_{n-1} of a group P with exponent p and nilpotency class $\leq n$. Let $P \leq \text{GL}(n+1, p)$ be the group of upper unitriangular matrices. For $x \in P$ we have

$$x^p - 1 = (x - 1)^p = (x - 1)^{n+1}(x - 1)^{p-n-1} = 0$$

since $p > n$. Hence, P has exponent p . For $i < j$, let $E_{ij} \in P$ be the unitriangular matrix with 1 on position (i, j) and zero elsewhere off the diagonal. A direct calculation shows that

$$[E_{ij}, E_{kl}] = E_{il}^{\delta_{jk}} E_{kj}^{-\delta_{il}}, \quad (3.2)$$

where $\delta_{jk}\delta_{il} = 0$ since $i < j$ and $k < l$. An induction shows that $P^{[k]}$ is generated by the matrices E_{ij} with $|j - i| \geq k$. In particular, $P^{[n]} = \langle E_{1,n+1} \rangle \cong C_p$ and $P^{[n+1]} = 1$ (see [15, Satz III.16.3]). So P has indeed nilpotency class n . We define $x_1 := E_{12}E_{n,n+1}$ and $x_i := E_{i,i+1}$ for $i = 2, \dots, n-1$. Then the right hand side of (3.1) is

$$[x_1, \dots, x_{n-1}, x_1] = \begin{cases} [x_1, \dots, x_{n-2}, E_{n-1,n+1}] = \dots = [x_1, E_{2,n+1}] = E_{1,n+1} & \text{if } n \geq 4, \\ [x_1, E_{24}E_{13}^{-1}] = [E_{12}E_{34}, E_{24}][E_{12}E_{34}, E_{13}]^{-1} = E_{14} & \text{if } n = 3. \end{cases}$$

Suppose first that $\pi(1) \neq 1$. Then x_1 appears only once in $c := [x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}]$. By Lemma 6, $c \equiv c_1 c_2 \pmod{\overline{F}^{[n+1]}}$, where c_1 and c_2 are n -fold commutators in the elements $E_{i,i+1}$. Both c_i contain $x_{\pi(1)}$ twice, so they must miss some $E_{r,r+1}$. But now each c_i lives inside a direct product of the form

$$Q := \left\{ \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} : Q_1 \leq \text{GL}(r, p), Q_2 \leq \text{GL}(n+1-r, p) \right\}.$$

Since Q has nilpotency class $< n$, we derive the contradiction $c \equiv c_1 c_2 \equiv 1 \pmod{\overline{F}^{[n+1]}}$.

Therefore, $\pi(1) = 1$. Now $[x_{\pi(n-1)}, x_1] \neq 1$ implies $\pi(n-1) \in \{2, n-1\}$ by (3.2). Assume first that $\pi(n-1) = n-1$. Then $[x_{\pi(n-2)}, x_{n-1}, x_1] = [x_{\pi(n-2)}, E_{n-1,n+1}] \neq 1$ implies $\pi(n-2) = n-2$. Inductively, one obtains $\pi = \text{id}$, $c = E_{1,n+1}$ and $a = 1$ in this case. Now suppose that $\pi(n-1) = 2$ and without loss of generality, $n \geq 4$. Here we use a different realization of \overline{F}_n inside P . More precisely, we reassign $x_i := E_{12}E_{23} \dots E_{n-1,n}$ for $i = 1, \dots, n-2$ and $x_{n-1} := E_{n,n+1}$. Then clearly, $c = [\dots, [x_1, x_1]] = 1$. On the other hand, the right hand side of (3.1) becomes

$$[x_1, \dots, x_{n-1}, x_1] = [x_1, \dots, x_1, E_{n-1,n+1}]^{-1} = \dots = [x_1, E_{2,n+1}]^{-1} = E_{1,n+1}^{-1}.$$

Contradiction. □

¹Since the proof of [6, Lemma 5] takes place in the non-nilpotent Lie algebra \mathfrak{gl}_n , it is not clear to me that the obtained result can actually be used to prove the main theorem.

We have duplicated x_1 in the commutator in Lemma 7 to avoid relations of the form

$$[* , \dots , * , x , y] \equiv [* , \dots , * , y , x]^{-1} \pmod{\overline{F}^{[n+1]}}.$$

To prove Cornulier's theorem, let $G = \{g_1, \dots, g_n\}$ be a finite group of order n . We construct a finite group H with $\text{Out}(H) \cong G$. Since $\text{Out}(1) = 1$ and $\text{Out}(C_3) \cong C_2$, we may assume that $n \geq 3$ (as in Lemma 7). We identify the generators x_i of \overline{F} with x_{g_i} and define

$$N := \langle [x_{hg_1}, \dots, x_{hg_{n-1}}, x_{hg_1}] : h \in G \rangle \overline{F}^{[n+1]} \leq \overline{F}^{[n]}.$$

Since $\overline{F}^{[n]}/\overline{F}^{[n+1]} \leq Z(\overline{F}_n)$, it follows that $N \leq \overline{F}$. Let $P := \overline{F}/N \cong \overline{F}_n/(N/\overline{F}^{[n+1]})$. Notice that P has exponent p and nilpotency class $\leq n$. Moreover, $P/P' \cong \overline{F}/\overline{F}' \cong C_p^n$. Again we will identify the x_i with their images in P .

Let $\mathbb{F}_p^\times = \langle \zeta \rangle$. For $1 \leq i \leq n$, the map $x_j \mapsto x_j^{\zeta^{\delta_{ij}}}$ can be extended to an automorphism q_i of \overline{F} . By Lemma 6,

$$q_i([x_{j_1}, \dots, x_{j_n}]) = [q_i(x_{j_1}), \dots, q_i(x_{j_n})] \equiv [x_{j_1}, \dots, x_{j_n}]^\gamma \pmod{\overline{F}^{[n+1]}}$$

for some $\gamma \in \mathbb{Z}$. In particular, $q_i(N) = N$ and q_i extends to an automorphism of P . Moreover, the group $Q := \langle q_1, \dots, q_n \rangle \leq \text{Aut}(P)$ is isomorphic to C_{p-1}^n . Finally, we define $H := P \rtimes Q$. As usual, we regard P and Q as subgroups of H . Then $q_i x_j q_i^{-1} = x_j^{\zeta^{\delta_{ij}}}$ for $1 \leq i, j \leq n$. Note that $H/P' \cong (C_p \rtimes C_{p-1})^n$.

The following result implies Cornulier's theorem.

Theorem 8. *With the notation above, $\text{Out}(H) \cong G$.*

Proof. For $h \in G$, the map $x_i \mapsto x_{hg_i}$ ($i = 1, \dots, n$) can be extended to an automorphism α_h of \overline{F} . By the definition of N , we have $\alpha_h(N) = N$. Therefore, we consider α_h as an automorphism of P . There is a similar automorphism $\beta_h \in \text{Aut}(Q)$ with $\beta_h(q_i) = q_{hg_i}$, where q_i is identified with q_{g_i} . Since

$$\alpha_h(q_i x_j q_i^{-1}) = \alpha_h(x_j)^{\zeta^{\delta_{ij}}} = x_{hg_j}^{\zeta^{\delta_{ij}}} = q_{hg_j} x_{hg_i} q_{hg_j}^{-1} = \beta_h(q_j) \alpha_h(x_i) \beta_h(q_j)^{-1},$$

the actions are compatible. This gives rise to a regular action $\alpha : G \rightarrow \text{Aut}(H)$. Since $g \neq 1$ acts non-trivially on $H/P \cong Q$, $\alpha(G) \cap \text{Inn}(H) = 1$. Thus, it suffices to show that $\text{Aut}(H) = \alpha(G) \text{Inn}(H)$.

To this end, let $\gamma \in \text{Aut}(H)$ be arbitrary. Then γ normalizes the normal Sylow p -subgroup P and P' . By Lemma 5, we may assume that γ permutes the factors of H/P' . So there exists a permutation $\pi \in S_n$ such that $\gamma(q_i) = q_{\pi(i)} \pmod{P'}$ and $\gamma(x_i) \equiv x_{\pi(i)} \pmod{P'}$ for $i = 1, \dots, n$. This implies

$$[x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}] = \gamma([x_1, \dots, x_{n-1}, x_1]) = \gamma(1) = 1$$

by Lemma 6. This yields an equation inside $N/\overline{F}^{[n+1]}$:

$$[x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}] \equiv \prod_{h \in G} [x_{hg_1}, \dots, x_{hg_{n-1}}, x_{hg_1}]^{a_h} \pmod{\overline{F}^{[n+1]}}$$

for some $0 \leq a_h \leq p-1$. By the universal property, this equation remains true when we set $x_{\pi(n)} = 1$. For the unique $h \in G$ with $x_{hg_n} = x_{\pi(n)}$ we deduce

$$[x_{\pi(1)}, \dots, x_{\pi(n-1)}, x_{\pi(1)}] \equiv [x_{hg_1}, \dots, x_{hg_{n-1}}, x_{hg_1}]^{a_h} \pmod{\overline{F}^{[n+1]}}.$$

By Lemma 7, $x_{\pi(i)} = x_{hg_i}$ for $i = 1, \dots, n-1$. Therefore, after composing γ with $\alpha(h)^{-1}$, we may assume that $\pi = 1$. Since $|P'|$ is coprime to $|Q|$ and $\gamma(Q) \leq P'Q$, there exists a $y \in P'$ with $\gamma(Q) = yQy^{-1}$ by the Schur–Zassenhaus theorem. Since conjugation with y does not affect H/P' , we may assume that $\gamma(Q) = Q$. In particular, γ centralizes Q .

Each quotient $P^{[k]}/P^{[k+1]}$ has a basis (as an elementary abelian group) consisting of some k -fold commutators in the x_i . By concatenation we obtain a basis c_1, \dots, c_s of $\times_{k=1}^n P^{[k]}/P^{[k+1]}$. For $c_i \in P^{[k]} \setminus P^{[k+1]}$, we have $q_j c_i q_j^{-1} \equiv c_i^{\zeta^l} \pmod{P^{[k+1]}}$ by Lemma 6, where l is the multiplicity of x_j as a component of c_i . Clearly, $l \leq k-1 \leq n-1 < p-1$. Hence, $\zeta^l \equiv 1 \pmod{p}$ can only hold if $l = 0$. This shows that q_j centralizes c_i if and only if x_j does not appear in c_i . Since the c_i form a basis, it follows that $C_P(q_j) = \langle x_i : i \neq j \rangle$. On the other hand, $q_j \gamma(x_i) q_j^{-1} = \gamma(q_j x_i q_j^{-1}) = \gamma(x_i)$ for $j \neq i$ shows that

$$\gamma(x_i) \in \bigcap_{j \neq i} C_P(q_j) = \langle x_i \rangle.$$

Since we already know that $\gamma(x_i) \equiv x_i \pmod{P'}$, we conclude $\gamma(x_i) = x_i$ for $i = 1, \dots, n$ and $\gamma = \text{id}$, as desired. \square

In order to estimate $|H|$ in terms of n , we consider the free nilpotent Lie algebra L over \mathbb{Q} of rank n and class n . By Witt's formula,

$$\dim L = \sum_{k=1}^n \frac{1}{k} \sum_{d|k} \mu(d) n^{k/d},$$

where μ is the Möbius function. This number grows roughly as n^{n-1} (see [2, Lemma 20.7]). Notice that $\mathbb{F}_p \otimes L$ is the corresponding free nilpotent Lie algebra over \mathbb{F}_p . Since $p > n$, the Lazard correspondence turns $\mathbb{F}_p \otimes L$ into \overline{F}_n (see [17, Example 10.24]). In particular, $|\overline{F}_n| = p^{\dim L}$. Moreover, an application of Lemma 7 reveals that $|N/\overline{F}^{[n+1]}| = p^n \sim (p-1)^n = |Q|$. Altogether, the order of magnitude of $|H|$ is $p^{n^{n-1}}$ (the estimate n^n in [6] is unjustified).

As a concrete example, consider $G \cong C_3$. Here we can take $p = 5$. Then $|\overline{F}_3| = 5^{14}$, $|P| = 5^{11}$ and $|H| = 2^6 5^{11}$. Cornulier's construction yields a group of order $2^6 5^{29}$, as he remarked at the end of [6].

One may ask if the group H can also be used to prove Pettet's theorem. This does not seem to be easy, since it is not clear whether every automorphism of $H \rtimes G$ normalizes H . Conversely, the group N in Theorem 4 has outer automorphisms, which do not come from G .

References

- [1] G. M. Bergman, *An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange*, Publ. Mat. **56** (2012), 91–126.
- [2] S. R. Blackburn, P. M. Neumann and G. Venkataraman, *Enumeration of finite groups*, Cambridge Tracts in Mathematics, Vol. 173, Cambridge University Press, 2007.
- [3] G. Braun and R. Göbel, *Outer automorphisms of locally finite p -groups*, J. Algebra **264** (2003), 55–67.
- [4] R. M. Bryant and L. G. Kovács, *Lie representations and groups of prime power order*, J. London Math. Soc. (2) **17** (1978), 415–421.

- [5] I. Bumagin and D. T. Wise, *Every group is an outer automorphism group of a finitely generated group*, J. Pure Appl. Algebra **200** (2005), 137–147.
- [6] Y. Cornulier, *Is every finite group the outer automorphism group of a finite group?*, MathOverflow, <https://mathoverflow.net/q/372563>.
- [7] M. Droste, M. Giraudet and R. Göbel, *All groups are outer automorphism groups of simple groups*, J. London Math. Soc. (2) **64** (2001), 565–575.
- [8] M. Dugas and R. Göbel, *Outer automorphisms of groups*, Illinois J. Math. **35** (1991), 27–46.
- [9] A. Entin and C. Tsang, *Normalizer Quotients of Symmetric Groups and Inner Holomorphs*, arXiv:2408.07133v1.
- [10] B. Fine and G. Rosenberger, *Number theory*, Birkhäuser/Springer, Cham, 2016.
- [11] R. Göbel and A. T. Paras, *Outer automorphism groups of metabelian groups*, J. Pure Appl. Algebra **149** (2000), 251–266.
- [12] B. Hartley and D. J. S. Robinson, *On finite complete groups*, Arch. Math. (Basel) **35** (1980), 67–74.
- [13] H. Heineken and H. Liebeck, *The occurrence of finite groups in the automorphism group of nilpotent groups of class 2*, Arch. Math. (Basel) **25** (1974), 8–16.
- [14] A. Hughes, *Automorphisms of nilpotent groups and supersolvable orders*, in: The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), 205–207, Proc. Sympos. Pure Math., Vol. 37, Amer. Math. Soc., Providence, RI, 1980.
- [15] B. Huppert, *Endliche Gruppen. I*, Grundlehren der Mathematischen Wissenschaften, Vol. 134, Springer-Verlag, Berlin, 1967.
- [16] B. Huppert and N. Blackburn, *Finite groups. II*, Grundlehren der Mathematischen Wissenschaften, Vol. 242, Springer-Verlag, Berlin, 1982.
- [17] E. I. Khukhro, *p -automorphisms of finite p -groups*, London Mathematical Society Lecture Note Series, Vol. 246, Cambridge University Press, Cambridge, 1998.
- [18] E. I. Khukhro and V. D. Mazurov, *The Kourovka notebook. Unsolved problems in group theory*, 18th edition, Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 2014.
- [19] S. Kojima, *Isometry transformations of hyperbolic 3-manifolds*, Topology Appl. **29** (1988), 297–307.
- [20] R. Lawton, *A note on a theorem of Heineken and Liebeck*, Arch. Math. (Basel) **31** (1978/79), 520–523.
- [21] W. Ledermann and B. H. Neumann, *On the order of the automorphism group of a finite group. I*, Proc. Roy. Soc. London. Ser. A. **233** (1956), 494–506.
- [22] T. Matumoto, *Any group is represented by an outerautomorphism group*, Hiroshima Math. J. **19** (1989), 209–219.
- [23] C. F. Miller and P. E. Schupp, *Embeddings into Hopfian groups*, J. Algebra **17** (1971), 171–176.
- [24] M. R. Pettet, *On inner automorphisms of finite groups*, Proc. Amer. Math. Soc. **106** (1989), 87–90.

- [25] M. R. Pettet, *Characterizing inner automorphisms of groups*, Arch. Math. (Basel) **55** (1990), 422–428.
- [26] B. Sambale, *On a theorem of Ledermann and Neumann*, Amer. Math. Monthly **127** (2020), 827–834.
- [27] P. E. Schupp, *A characterization of inner automorphisms*, Proc. Amer. Math. Soc. **101** (1987), 226–228.
- [28] U. H. M. Webb, *The occurrence of groups as automorphisms of nilpotent p -groups*, Arch. Math. (Basel) **37** (1981), 481–498.